Lecture Notes for ORIE 473: Empirical Methods in Financial Engineering

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Preface

These notes are intended for an advanced undergraduate or masters level course on mathematical modeling in finance and financial engineering. I use these notes to teach the course OR & IE 473 intended for majors in operations research and industrial engineering. Many of our majors go to to careers in banking and finance and our master of engineering (MENG) program has a popular financial engineering option.

The prerequisites for my course are two years of engineering mathematics including matrix algebra and multivariate calculus plus a year of probability and statistics, all required course in the OR&IE major.

The emphasis in these notes is on empirical research methods, that is, data analysis and statistical inference.

The course has several goals:

- To reinforce the material taught in the prerequisite courses in probability and statistics by illustrating the main concepts of probability and statistics with concrete examples from finance
- To serve as a capstone course integrating statistics, probability, and to some extent optimization
- To introduce undergraduates to finance and financial engineering
- To help students appreciate the role of empirical research in finance and in operations research
- To expose students to new statistical methods: time series, GARCH models, resampling (bootstrap), and nonparametric regression
- To teach the use of the MATLAB, SAS, and MINITAB software packages

My colleage at Cornell, Marty Wells, created OR & IE 473. It was his insight that financial engineers needed a better appreciation of statistical

methods and that perhaps only a statistician could design a course to serve this need. Marty taught the course for two years, and then I took over when Marty became involved in other projects. As i prepared to teach OR & IE 473 for the first time, I realized that there were no textbooks at the undergraduate level that I could use. I decided to type rough notes and put them on a course web site. The notes have been expanded and been refined in subsequent years.

I have found teaching OR & IE 473 to be very rewarding. Students like to see "real data." Finance is an excellent source of examples for a statistics course, and financial data are readily accessable from many web sites. Most students have some familiarity with the stock market and are eager to learn more.

Chapter 1

Introduction: 1/4/02

1.1 About these notes

These are notes for OR&IE 473 at Cornell University. This course is intended for juniors, seniors, and masters students who have taken courses in probability, statistics, and stochastic processes[1].

There is no single textbook that covers all of the topics of this course. That is the reason why these notes were written. However, there are a number of textbooks that cover the material in one or more chapters of these notes. These books are on reserve in the Engineering Library. See the links to "Books on Reserve" and "Suggested Supplemental Reading" on the course's web site.

These notes are being updated frequently. After the title of each chapter you will find the date when that chapter was last edited. I recommend that you print out a chapter just before you plan to read it, so that you'll have the most recent edition.

1.2 What this course is about

The title of this course is "Empirical Research Methods in Financial Engineering."

"Empirical" means derived from experience, observation, or experi-

[1]The course in stochastic processes may be taken concurrently.

ment — so we are going to work with data. We'll be doing statistics.

Financial engineering is the construction of financial products such as stock options.

Finance makes extensive use of probability models, e.g., those used to derive the famous Black-Scholes formula.

- are these models supported by financial markets data?
- how are the parameters in these models estimated?

Let's look ahead to the Black-Scholes formula for the price of a European call option. "Now" is called time 0. The maturity date of the option is T. The option gives us the right, but *not* the obligation, to purchase one share of stock for E dollars at time T. Let S_T be the price of the stock at time T. At time 0, T and E are known but S_T is unknown.

At time T, S_T will become known. If at time T we learn that $S_T > E$ then we will exercise the option and purchase one share. We can immediately sell the share for S_T dollars and earn a profit of $S_T - E$ dollars, less the cost of the option.

If at time T, $S_T < E$ then we do not exercise the option. The option expires and we lose the original cost of the option, but no more.

The value of the option at time T is, therefore, $\max\{0,\ S - E\}$.[2] But right now at time 0, what is the value of the option, i.e., the price for which it should sell on the market?

Prior to the 1970's, options were priced by "seat of pants". Then Black, Scholes, and Merton deduced the correct price of a call option from a mathematical model (and much hard thinking).

They assumed that one can lend and borrow at a risk-free rate r. Thus, if B_t is the price at time t of a risk-free bond purchased for \$1 at time 0, then $B_0 = 1$ and

$$B_t = B_0 \exp(rt) = \exp(rt).$$

[2]The max of a set of numbers is the largest number in the set. Therefore, $\max\{0,\ S - E\}$ equal 0 if $S - E$ is negative and equals $S - E$ if $S - E$ is positive

Let S_t be the price of the underlying stock. They assumed that

$$S_t = S_0 \exp(\mu t + \sigma W_t),$$

where μ is a "drift" or "growth rate," W_t is a Brownian motion stochastic process, and σ is a standard deviation that measures the volatility of the stock. In this course, you will learn exactly what this model means. Right now, the "take home" message is that there are precise mathematical models of stock price movements that we can check against the data. Also, there are important parameters such as μ and σ that must be estimated from data.

The Black-Scholes formula is

$$C = \Phi(d_1)S_0 - \Phi(d_2)E\exp(-rT)$$

where C is the price of the option at time 0, Φ is the standard normal CDF,

$$d_1 = \frac{\log(S_0/E) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

The formula is, quite obviously, complicated and it not easy to derive, but it is easy to compute and was hard-wired into calculators almost immediately after it was discovered; the Black-Scholes formula and hand-held calculators both emerged in the early 1970's.

We will be interested in the underlying assumptions behind the formula. Remember: GI – GO (garbage in, garbage out)! If the assumptions don't hold, then there is no reason to trust the Black-Scholes formula, despite the impressive mathematics behind it.

The equation $B_t = \exp(rt)$ of continuous compounding is the solution to the differential equation

$$\frac{d\,B_t}{d\,t} = rB_t.$$

The general solution is $B_t = B_0\exp(rt)$ and $B_0 = 1$ since we have assumed that the bond can be purchased for \$1 at time 0.

Where does

$$S_t = S_0\exp(\sigma W_t + \mu t)$$

come from? If σ were 0, then this would be exponential growth, $S_t = S_0\exp(\mu t)$, just like the bond price B_t. The term σW_t comes from the random behavior of stock prices. σ is a standard deviation, essentially of the

changes in the stock prices. The random process W_t is something we will need to learn much more about; and we will.

In this course we will

- study models fo financial markets (they are complex but fascinating)
- learn to test the models — do they fit financial markets data adequately?
- estimate parameters in the models such as μ and σ that are essential for correct pricing of financial products such as a call option.

Key question: How do the prices of stocks and other financial assets behave?

Looking ahead to where this course is going

- We will start by defining "returns" on the prices of a stock
- We will then look at "ARIMA models"
  - these are models for "time series," which are sequences of data sampled over time
  - ARIMA models are stochastic processes
- After looking at returns and time series models of return behavior we will look at optimal portfolios of risky assets (e.g., stocks) and of risky assets and risk-free bonds (e.g., US Treasury bills).
  - This will take us to the famous Capital Asset Pricing Model (CAPM)

Looking even farther ahead, we will later return to the pricing of stock options by the Black-Scholes formula and cover other areas of financial engineering such as the term structure of interest rates.

But before we get into applications of probability and statistics in financial engineering, we need to review some probability and statistics so that we are all up to speed.

Chapter 2

Review of Probability and Statistics: 5/13/02

2.1 Some Basic Definitions

Random variables — a random variable has a large set of possible values but only one will actually occur. The set of possible values and their probabilities are called the probability distribution of the random variable.

Continuous random variable — X is a continuously distributed random variable if it has a probability density function (pdf) f_X such that

$$P(X \in A) = \int_A f_X(x)dx \quad \text{for all sets} \quad A.$$

CDFs — The cumulative distribution function (CDF) of X is[1]

$$F_X(x) := P(X \leq x)$$

If X has a pdf then

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du.$$

Quantiles — if the CDF of X is continuous and strictly increasing then it has a inverse function F^{-1}. For each q between 0 and 1, $F^{-1}(q)$ is called the qth quantile or 100qth percentile. The probability that X is below its qth quantile is precisely q. The median is the 50% percentile or .5 quantile. The 25% and 75% percentiles (.25 and .75 quantiles) are called the first and third quartiles.

[1] In these notes, ":=" means "equal by definition".

Figure 2.1 illustrates quantiles. The top, left plot has a continuous, strictly increasing CDF. There is a unique qth quantile $F^{-1}(q)$ and $P(X < F^{-1}(q)) = P(X \leq F^{-1}(q)) = q$. The top, right plot has a discrete CDF. There is a unique qth quantile but $P(X < F^{-1}(q)) < q < P(X \leq F^{-1}(q))$. The bottom, left plot has a continuous, but not strictly increasing CDF. There is an interval of qth quantiles and each satisfies $P(X < F^{-1}(q)) = P(X \leq F^{-1}(q)) = q$.

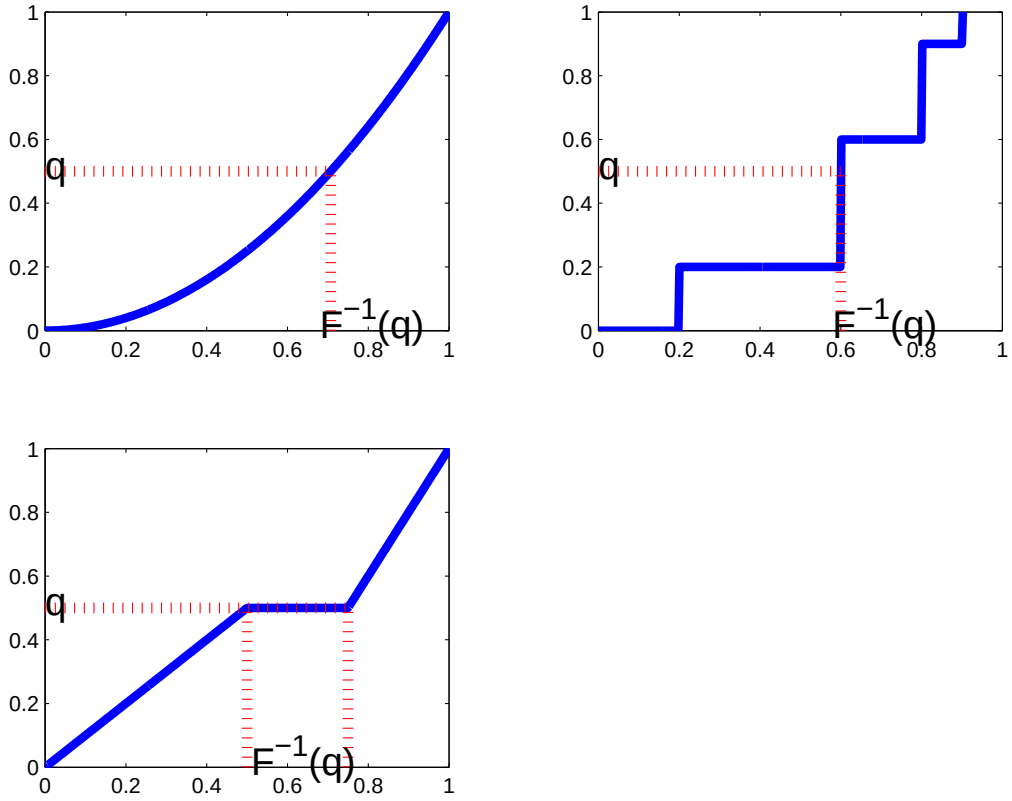


Figure 2.1: *Quantiles*

Expectations and Variances — The expectation of X is

$$E(X) := \int_{-\infty}^{+\infty} x f_X(x) dx.$$

The variance of X is

$$\sigma_X^2 := \int \{x - E(X)\}^2 f_X(x) dx = E\{X - E(X)\}^2.$$

Useful formula: $\sigma_X^2 = E(X^2) - \{E(X)\}^2$.
The standard deviation is the square root of the variance:

$$\sigma_X := \sqrt{E\{X - E(X)\}^2}.$$

If $X_1, \ldots, X_n$ is a sample from a probability distribution, then the expectation can be estimated by the sample mean

$$\overline{X} = \sum_{i=1}^{n} X_i \tag{2.1}$$

and the variance can be estimated by the sample variance

$$s_X^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1} \tag{2.2}$$

Bivariate distributions — A pair of continuously distributed random variables, (X, Y), has a bivariate density $f_{XY}(x, y)$:

$$P\{(X, Y) \in A\} = \int \int_A f(x, y) dx dy.$$

Correlation and Covariance —

$$\sigma_{XY} = E\Big[\{X - E(X)\}\{Y - E(Y)\}\Big].$$

If (X, Y) are continuously distributed, then

$$\sigma_{XY} = \int \{x - E(X)\}\{y - E(Y)\} f_{XY}(x, y)\, dx\, dy.$$

Useful formulas:

$$\begin{aligned}
\sigma_{XY} &= E(XY) - E(x)E(y) && (2.3)\\
\sigma_{XY} &= E[\{X - E(X)\}Y] && (2.4)\\
\sigma_{XY} &= E[\{Y - E(Y)\}X] && (2.5)\\
\sigma_{XY} &= E(XY) \text{ if } E(X) = 0 \text{ or } E(Y) = 0 && (2.6)
\end{aligned}$$

The correlation coefficient between X and Y is $\rho_{XY} := \sigma_{XY}/\sigma_X\, \sigma_Y$.

Given a bivariate sample $\{(X_i, Y_i\}_{i=1}^n$, the sample correlation coefficient is

$$\frac{\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y})}{s_Y s_Y} \tag{2.7}$$

where $\overline{X}$ and $\overline{Y}$ are the sample means and s_X and s_Y are the sample standard deviations.

Figure 2.2 gives scatterplots and the sample correlation coefficients for eight random samples. Notice that

- an absolute correlation of .25 is very weak
- an absolute correlation of .5 is only moderate
- an absolute correlation of .95 is rather strong
- an absolute correlation of 1 implies a linear relationship
- a strong nonlinear relationship may or may not imply a high correlation
- positive correlations imply an increasing relationship (as X increases, Y increases on average)
- negative correlations imply an increasing relationship (as X increases, Y decreases on average)

Independence — X and Y are independent if for all sets A and B,

$$P(X \in A \text{ and } Y \in B) = P(X \in A)\, P(Y \in B).$$

If X and Y are independent then for all functions g and h,

$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}.$$

This fact can be used to prove that if X and Y are independent, then $\sigma_{XY} = 0$.

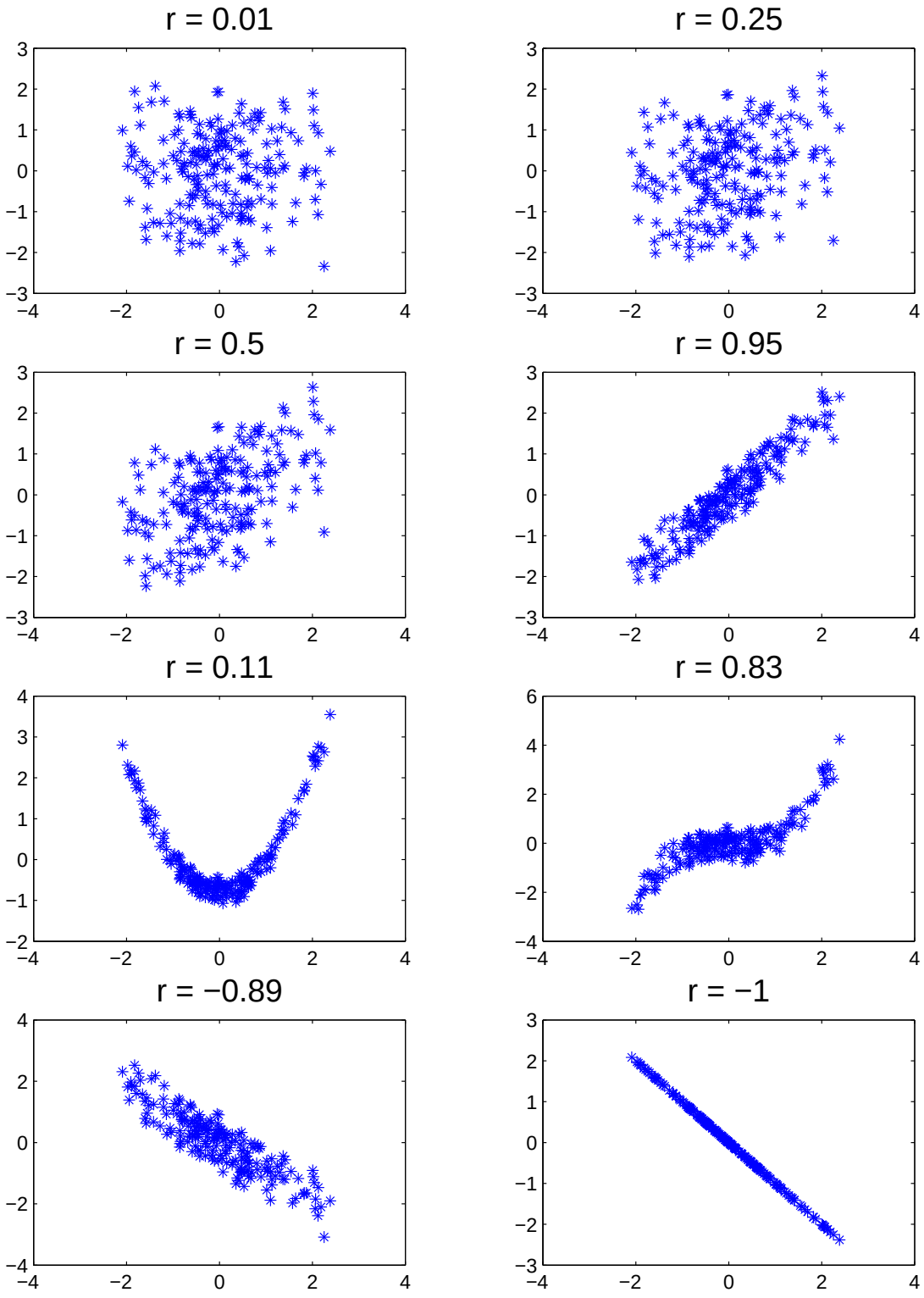


Figure 2.2: *Sample correlation coefficients for eight random samples.*

2.2 Best Linear Prediction

Suppose we observe X and we want to predict Y which has not yet been observed. If ρ_{XY} is not zero, then knowing X can help us predict Y.

A linear predictor of Y based on X is a function $\beta_0 + \beta_1 X$ where β_0 and β_1 are parameters that we can choose. Best linear prediction means finding β_0 and β_1 so that expected squared prediction error given by

$$E\{Y - (\beta_0 + \beta_1 X)\}^2$$

is minimized.

$$E\{Y - (\beta_0 + \beta_1 X)\}^2 = E(Y^2) - 2\beta_0 E(Y) - 2\beta_1 E(XY) + (\beta_0 + \beta_1 X)^2.$$

Setting the partial derivatives to zero we get

$$\begin{aligned} 0 &= -E(Y) + \beta_0 + \beta_1 E(X) \text{ and} \\ 0 &= -E(XY) + \beta_0 E(X) + \beta_1 E(X^2). \end{aligned}$$

After some algebra (exercise) we find that

$$\beta_1 = \sigma_{XY}/\sigma_X^2 \tag{2.8}$$

and

$$\beta_0 = E(Y) - \beta_1 E(X) = E(Y) - \sigma_{XY}/\sigma_X^2\, E(X). \tag{2.9}$$

Thus, the best linear predictor of Y is

$$\widehat{Y} := \beta_0 + \beta_1 X = E(Y) + (\sigma_{XY}/\sigma_X^2)\{X - E(X)\} \tag{2.10}$$

2.2.1 Prediction Error

The prediction error is $Y - \widehat{Y}$. It is easy to prove that $E\{Y - \widehat{Y}\} = 0$ so that the prediction is "unbiased." With a little algebra we can show that the expected squared prediction error is

$$E\{Y - \widehat{Y}\}^2 = \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2} = \sigma_Y^2(1 - \rho_{XY}^2). \tag{2.11}$$

How much does X help us predict Y? To answer this question, notice first that if we do not observe X then we must predict Y using a constant, which we will denote by c. The expected squared prediction error is $E(Y - c)^2$. Some algebra shows that

$$E(Y - c)^2 = \text{Var}(Y) + \{c - E(Y)\}^2, \tag{2.12}$$

which shows that the expected squared prediction error is minimized by $c = E(Y)$. Thus, the best predictor of Y is $E(Y)$ and the expected squared prediction error is σ_Y^2. When X is observed, then the expected squared prediction error is $\sigma_Y^2(1 - \rho_{XY}^2)$. Therefore, ρ_{XY}^2 is the fraction by which the prediction error is reduced when X is known. This is an important fact that we will see again.

Example: If $\rho_{XY} = .5$, then the prediction error is reduced by 25% by observing X. If $\sigma_Y^2 = 3$, then the expected squared prediction error is 3 if X is unobserved but only $2.25 = 3\{1 - (.5)^2\}$ if X is observed.

2.3 Conditional Distributions

Let $f_{XY}(x, y)$ be the joint density of a pair of random variables, (X, Y).

The marginal density of X is $f_X(x) := \int f_{XY}(x, y)dy$ and similarly for f_Y.

The conditional density of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

The conditional expectation of Y given X is just the expectation calculated using $f_{Y|X}(y|x)$:

$$E(Y|X = x) = \int y f_{Y|X}(y|x)dy$$

which is, of course, a function of x.

The conditional variance of Y given X is

$$\text{Var}(Y|X = x) = \int \{y - E(Y|X = x)\}^2 f_{Y|X}(y|x)dy.$$

Example: Suppose $f_{XY}(x, y) = 2$ on $0 < x < 1$ and $x < y < 1$.

Then the marginal density of X is $f_X(x) = 2(1 - x)$.

The conditional density of Y given X is $f_{Y|X}(y|x) = (1 - x)^{-1}$ for $x < y < 1$.

The conditional expectation of Y is

$$E(Y|X = x) = \frac{1 + x}{2}.$$

The conditional variance of Y is

$$\text{Var}(Y|X = x) = \frac{(1 - x)^2}{12}.$$

2.4 The Normal Distribution

The standard normal distribution has density

$$\phi(x) := \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right).$$

The $N(\mu, \sigma^2)$ density is

$$\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right).$$

The standard normal CDF is

$$\Phi(x) := \int_{-\infty}^{x} \phi(u)du.$$

Φ can be evaluated using tables (ugh!) or more easily using software such as MATLAB or MINITAB.

If $X \sim N(\mu, \sigma^2)$[2] then $P(X \leq x) = \Phi\{(x-\mu)/\sigma\}$.

Example: If $X \sim N(5, 4)$ then what is $P(X \leq 7)$. *Answer:* Using $x = 7$, $\mu = 5$, and $\sigma^2 = 4$, we have $(x-\mu)/\sigma = (7-5)/2 = 1$ and then $\Phi(1) = .8413$. In MATLAB, "`cdfn(1)`" gives "`ans = 0.8413`".

2.4.1 Conditional expectations and variance

The calculation of conditional expectations and variances can be difficult for some probability distributions, but it is quite easy for a pair (X, Y) that has a bivariate normal distribution.

For a bivariate normal pair, the conditional expectation of Y given X equals the best linear predictor of Y given X:

$$E(Y|X) = E(Y) + \frac{\sigma_{XY}}{\sigma_X^2}\{X - E(X)\}.$$

The conditional variance of Y given X is the expected squared prediction error:

$$\text{Var}(Y|X) = \sigma_Y^2(1 - \rho_{XY}^2)$$

[2] In these notes, "$\sim$" means "distributed as."

2.5 Linear Functions of Random Variables

$$E(aY + b) = aE(Y) + b$$

where Y is a random variable and a and b are constants. Also,

$$\text{Var}(aY + b) = a^2\text{Var}(Y).$$

If X and Y are random variables and w_1 and w_2 are constants, then

$$E(w_1X + w_2Y) = w_1E(X) + w_2E(Y),$$

and

$$\text{Var}(w_1X + w_2Y) = w_1^2\text{Var}(X) + 2w_1w_2\text{Cov}(X, Y) + w_2^2\text{Var}(Y).$$

Check that

$$\text{Var}(w_1X + w_2Y) = \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \text{Var}(X) & \text{Cov}(X, Y) \\ \text{Cov}(X, Y) & \text{Var}(Y) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Let $\boldsymbol{X} = (X_1, \ldots, X_N)^\mathsf{T}$ be a random vector. We define the expectation vector of $\boldsymbol{X}$ to be

$$\begin{pmatrix} E(X_1) \\ \vdots \\ E(X_N) \end{pmatrix}.$$

The covariance matrix of $\boldsymbol{X}$ is

$$\text{COV}(\boldsymbol{X}) := \begin{pmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_N) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) & \cdots & \text{Cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_N, x_1) & \text{Cov}(X_N, X_2) & \cdots & \text{Var}(X_N) \end{pmatrix}.$$

Let $\boldsymbol{w} = (w_1, \ldots, w_N)^\mathsf{T}$ be a vector of weights. Then

$$\boldsymbol{w}^\mathsf{T}\boldsymbol{X} = \sum_{i=1}^{N} w_iX_i$$

is a weighted average of the components of bX; it is a random variable.

One can show that

$$E(\boldsymbol{w}^\mathsf{T}\boldsymbol{X}) = \boldsymbol{w}^\mathsf{T}\{E(\boldsymbol{X})\}.$$

Also

$$\text{Var}(\boldsymbol{w}^\mathsf{T}\boldsymbol{X}) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i\, w_j\, \text{Cov}(X_i, X_j)$$

This result can be expressed more simply using vector/matrix notation:

$$\text{Var}(\boldsymbol{w}^\mathsf{T}\boldsymbol{X}) = \boldsymbol{w}^\mathsf{T}\text{COV}(\boldsymbol{X})\boldsymbol{w}. \tag{2.13}$$

Example: Suppose that $\boldsymbol{X} = (X_1\ X_2\ X_3)^\mathsf{T}$, $\text{Var}(X_1) = 2$, $\text{Var}(X_2) = 3$, $\text{Var}(X_3) = 5$, $\rho_{X_1,X_2} = .6$, and that X_1 and X_2 are independent of X_3. Find $\text{Var}(X_1 + X_2 + \frac{1}{2}X_3)$.

Answer: The covariance between X_1 and X_3 is 0 by independence and the same is true of X_2 and X_3. The covariance between X_1 and X_2 is $(.6)\sqrt{(2)(3)} = 1.47$. Therefore,

$$\text{COV}(\boldsymbol{X}) = \begin{pmatrix} 2 & 1.47 & 0 \\ 1.47 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

and

$$\begin{aligned} \text{Var}(X_1 + X_2 + X_3/2) &= \begin{pmatrix} 1 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1.47 & 0 \\ 1.47 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3.47 \\ 4.47 \\ 2.5 \end{pmatrix} \\ &= 9.19. \end{aligned}$$

Two weighted averages: More generally, if $\boldsymbol{w}_1^\mathsf{T}\boldsymbol{X}$ and $\boldsymbol{w}_2^\mathsf{T}\boldsymbol{X}$ are two weighted averages of the components of $\boldsymbol{X}$, then

$$\text{Cov}(\boldsymbol{w}_1^\mathsf{T}\boldsymbol{X}, \boldsymbol{w}_2^\mathsf{T}\boldsymbol{X}) = \boldsymbol{w}_1^\mathsf{T}\text{COV}(\boldsymbol{X})\boldsymbol{w}_2. \tag{2.14}$$

Example continued: Suppose as in the previous example that

$$\boldsymbol{X} = (\, X_1 \quad X_2 \quad X_3\,)^\mathsf{T},$$

$\text{Var}(X_1) = 2$, $\text{Var}(X_2) = 3$, $\text{Var}(X_3) = 5$, $\rho_{X_1,X_2} = .6$, and that X_1 and X_2 are independent of X_3. Find the covariance between $(X_1 + X_2 + X_3)/3$ and $(X_1 + X_2)/2$.

Answer: Let

$$\boldsymbol{w}_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

and

$$\boldsymbol{w}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

Then

$$\begin{aligned} \text{Cov}\left\{\frac{X_1+X_2}{2}, \frac{X_1+X_2+X_3}{3}\right\} &= \boldsymbol{w}_1^{\mathsf{T}}\text{COV}(\boldsymbol{X})\boldsymbol{w}_2 \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & 1.47 & 0 \\ 1.47 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1.157 & 1.490 & 1.667 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \\ &= 1.323. \end{aligned}$$

Independence and variances of sums: If $X_1, \ldots, X_n$ are independent, or at least uncorrelated, then

$$\text{Var}(\boldsymbol{w}^{\mathsf{T}}\boldsymbol{X}) = \sum_{i-1}^{N} w_i^2 \text{Var}(X_i). \tag{2.15}$$

Important fact: If $\boldsymbol{X}$ has a multivariate normal distribution, then $\boldsymbol{w}^{\mathsf{T}}\boldsymbol{X}$ is a normally distributed random variable.

Example: Suppose that $E(X_1) = 1$, $E(X_2) = 1.5$, $\sigma^2_{X_1} = 1$, $\sigma^2_{X_2} = 2$, and $\text{Cov}(X_1, X_2) = .5$. Find $E(.3X_1 + .7X_2)$ and $\text{Var}(.3X_1 + .7X_2)$. If $(X_1\ X_2)^{\mathsf{T}}$ is bivariate normal, find $P(.3X_1 + .7X_2 < 2)$.

Answer: $E(.3X_1 + .7X_2) = 1.35$, $\text{Var}(.3X_1 + .7X_2) = 1.28$, and $P(.3X_1 + .7X_2 < 2) = \Phi\{(2 - 1.35)/\sqrt{1.28}\} = \Phi(.5745) = .7172$.

2.6 Hypothesis Testing

Statistical hypothesis testing uses data to test whether a certain hypothesis, called the null hypothesis, is true or not. The negation of the null hypothesis is called the alternative hypothesis. For example, suppose that $X_1, \ldots, X_N$ are IN$(\mu, 1)$[3] and μ is unknown. The null hypothesis could be that μ is 1. Then we write H_0: $\mu = 1$ and H_1: $\mu \neq 1$ to denote the null and alternative hypotheses.

[3]The notation "$X_1, \ldots, X_N$ are IN(μ, σ^2)" means that the X_i are independent and that each X_i is $N(\mu, \sigma^2)$.

The rejection region is the set of possible samples that lead us to reject H_0. For example, we might reject H_0 if $|\overline{X} - 1|$ exceeds some cutoff value c.

There are two types of errors that we hope to avoid. If the null hypothesis is true but we reject it, then we are making a type I error. Conversely, if the null hypothesis is false and we accept it then we are making a type II error.

The rejection region is chosen to keep the probability of a type I error below a pre-specified small value called the level of the test and often denoted by α. Typical values of α used in practice are .01, .05, or .1. As α is made smaller, the rejection region must be made smaller. For example, if we reject the null hypothesis that $\mu = 1$ when $|\overline{X} - 1|$ exceeds c then c gets larger as the α gets smaller.

2.6.1 P-values

Rather that specifying α and decided whether to accept or reject the null hypothesis at that α, we might ask "for what values of α do we reject the null hypothesis?" The *p-value* for a sample is defined as the smallest value of α for which the null hypothesis is rejected for that sample. In other words, to perform the test using a given sample we first find the p-value of that sample and then H_0 is rejected if we decide to use α larger than the p-value and H_0 is accepted if we use α smaller than the p-value. Thus, a small p-value is evidence against the null hypothesis while a large p-value shows that the data are consistent with the null hypothesis.

Example: If the p-value of a sample is .0331, then we reject H_0 if we useα equal to .05 or .1 but we accept H_0 we use $\alpha = .01$.

The p-value not only tells us whether the null hypothesis should be accepted or rejected, but it also tells us whether or not the decision to accept or reject H_0 is a close call. For example, if we are using $\alpha = .05$ and the p-value were .0473, then we would reject H_0 but we would know the decision was close. If instead the p-value were .00107, then we would know the decision was not so close.

When performing hypothesis tests, statistical software routinely calculates p-values. Doing this is much more convenient that asking the user to specify α and then reporting whether the null hypothesis is accepted or rejected for that α.

2.7 Maximum Likelihood Estimation

Maximum likelihood is the most important and widespread method of estimation. Many well-known estimators such as the sample mean and the least-squares estimator in regression are maximum likelihood estimators. Maximum likelihood is a very useful in practice and tends to give more precise estimates than other methods of estimation.

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^\mathsf{T}$ be a vector of data and let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)^\mathsf{T}$ be a vector of parameters. Suppose that $f(\boldsymbol{y}; \boldsymbol{\theta})$ is the density of $\boldsymbol{Y}$ which depends on the parameters.

Example: Suppose that $Y_1, \ldots, Y_n$ are IID $N(\mu, \sigma^2)$. Then $\boldsymbol{\theta} = (\mu, \sigma^2)$. Also,

$$f(\boldsymbol{y}; \boldsymbol{\theta}) = \prod_{i=1}^{n} \frac{1}{\sigma} \phi\left(\frac{Y_i - \mu}{\sigma}\right) = \frac{1}{\sigma^n (2\pi)^{n/2}} \exp\left\{\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - \mu)^2\right\}.$$

$L(\boldsymbol{\theta}) := f(\boldsymbol{Y}; \boldsymbol{\theta})$ is called the "likelihood function" and is the density evaluated at the observed data. It tells us the likelihood of the sample that was actually observed as a function of $\boldsymbol{\theta}$. The maximum likelihood estimator (MLE) is the value of $\boldsymbol{\theta}$ that maximizes the likelihood function. In other words, the MLE is the value of $\boldsymbol{\theta}$ that maximizes the likelihood of the data that was observed. We will denote the MLE by $\widehat{\boldsymbol{\theta}}_{ML}$. Often it is mathematically easier to maximize $\log\{L(\boldsymbol{\theta})\}$; since the log function is increasing, maximizing $\log\{L(\boldsymbol{\theta})\}$ is equivalent to maximizing $L(\boldsymbol{\theta})$.

Example: In the example above, it is an easy calculus exercise to show that $\widehat{\mu}_{ML} = \overline{Y}$. Also, with μ fixed at its MLE, the MLE of σ^2 solves

$$0 = \frac{d}{d\,\sigma^2} \log\{L(\widehat{\mu}_{ML}, \sigma^2)\} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (Y_i - \widehat{\mu}_{ML})^2.$$

The solution to this equation is

$$\widehat{\sigma}^2_{ML} = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \overline{Y})^2.$$

The MLE of σ^2 has a small bias. The "bias-corrected" MLE is the so-called "sample variance" defined before as

$$s^2_Y = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 \tag{2.16}$$

In a "textbook example" such as the one above, it is possible to find an explicit formula for the MLE. With more complex models, there is no explicit formula for the MLE. Rather, one writes a program to compute $\log\{L(\boldsymbol{\theta})\}$ for any value of $\boldsymbol{\theta}$ and then uses optimization software to maximize this function numerically. For some models such as the ARIMA time series models discussed in Chapter 4, there are software packages, e.g, MINITAB and SAS, that compute the MLE; the computation of the log-likelihood function has been pre-programmed.

2.8 Likelihood Ratio Tests

Likelihood ratio tests, like maximum likelihood estimation, are a convenient, all-purpose tool. We will consider likelihood ratio tests when we wish to test a restriction on a subset of the parameters. Let

$$\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \end{pmatrix}$$

be a partitioning of the parameter vector into two component vectors. Suppose we want to test a hypothesis about $\boldsymbol{\theta}_1$ without making any hypothesis about the value of $\boldsymbol{\theta}_2$. For example, we might want to test that a population mean is zero; then $\boldsymbol{\theta}_1 = \mu$ and $\boldsymbol{\theta}_2 = \sigma^2$.

Let $\boldsymbol{\theta}_{1,0}$ be the hypothesized value of $\boldsymbol{\theta}_1$, e.g., $\boldsymbol{\theta}_{1,0} = 0$ if we want to test that μ is zero. Then the hypotheses are

$$\mathrm{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_{1,0} \quad \text{and} \quad \mathrm{H}_1 : \boldsymbol{\theta}_0 \neq \boldsymbol{\theta}_{1,0}.$$

Notice that neither hypothesis says anything about $\boldsymbol{\theta}_2$. For example, if we are testing that μ is zero then the hypotheses are

$$\mathrm{H}_0 : \mu = 0 \quad \text{and} \quad \mathrm{H}_1 : \mu \neq 0,$$

and neither hypothesis specifies anything about σ.

Let $\widehat{\boldsymbol{\theta}}_{ML}$ be the maximum likelihood estimator and let $\widehat{\boldsymbol{\theta}}_{2,0}$ be the value of $\boldsymbol{\theta}_2$ that maximizes $L(\boldsymbol{\theta})$ when $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_{1,0}$.

Idea: If H_0 is true, then $L(\boldsymbol{\theta}_{1,0}, \widehat{\boldsymbol{\theta}}_{2,0})$ should be similar to $L(\widehat{\boldsymbol{\theta}})$. Otherwise, $L(\boldsymbol{\theta}_{1,0}, \widehat{\boldsymbol{\theta}}_{2,0})$ should be *smaller* that $L(\widehat{\boldsymbol{\theta}})$.

The likelihood ratio test rejects H_0 if

$$2\Big[\log\{L(\widehat{\boldsymbol{\theta}}_{ML})\} - \log\{L(\boldsymbol{\theta}_{1,0}, \widehat{\boldsymbol{\theta}}_{2,0})\}\Big] \geq \chi^2_{\alpha;\,\dim(\boldsymbol{\theta}_1)}.$$

Here $\dim(\boldsymbol{\theta}_1)$ is the dimension (number of components) of $\boldsymbol{\theta}_1$ and $\chi^2_{\alpha,k}$ is the α upper-probability value of the chi-squared distribution with k degrees of freedom. In other words, $\chi^2_{\alpha,k}$ is the $(1-\alpha)$ quantile of the chi-squared distribution with k degrees of freedom so that the probability above $\chi^2_{\alpha,k}$ is α.

Example: Suppose again that $Y_1, \ldots, Y_n$ are IID $N(\mu, \sigma^2)$ and $\boldsymbol{\theta} = (\mu, \sigma^2)$. We want to test that μ is zero. Note that

$$\log(L) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \mu)^2.$$

If we evaluate $\log(L)$ at the MLE, we get

$$\log\{L(\overline{Y}, \widehat{\sigma}^2_{ML})\} = -\frac{n}{2}\{1 + \log(2\pi) + \log(\widehat{\sigma}^2_{ML})\}.$$

The value of σ^2 that maximizes L when $\mu = 0$ is

$$\widehat{\sigma}^2_0 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2.$$

Therefore,

$$2\Big[\log\{L(\overline{Y}, \widehat{\sigma}^2_{ML})\} - \log\{L(0, \widehat{\sigma}^2_0)\}\Big] = n\log\left(\frac{\widehat{\sigma}^2_0}{\widehat{\sigma}^2_{ML}}\right) = n\log\left(\frac{\sum_{i=1}^n Y_i^2}{\sum_{i=1}^n (Y_i - \overline{Y})^2}\right).$$

The likelihood ratio test rejects H_0 if

$$n\log\left(\frac{\sum_{i=1}^n Y_i^2}{\sum_{i=1}^n (Y_i - \overline{Y})^2}\right) > \chi^2_{\alpha,1}. \tag{2.17}$$

To appreciate why (2.17) is a reasonable test consider first what happens if $\mu = 0$. Then $\overline{Y}$ will be close to 0 and fraction inside the log in (2.17) will be close to 1. The log of 1 is 0 so the left hand side of (2.17) will tend to be small so that we do not reject the null hypothesis. Of course, this is the right decision if $\mu = 0$. Simple algebra shows that

$$\sum_{i=1}^{n} Y_i^2 = \sum_{i=1}^{n}(Y_i - \overline{Y})^2 + n(\overline{Y})^2.$$

Therefore, if μ is not 0 then the left hand side of (2.17) will tend to be large so that we reject the null hypothesis as we should.

2.9 Summary

- random variable can be described by
  - probability distribuiton
  - pdf
  - CDF
- probability distributions can be summarized by
  - expected value
  - variance (σ^2) or standard deviation (σ)
  - covariance and correlation
- for q between 0 and 1, $F^{-1}(q)$ is called the q**th quantile** or **100**q**th percentile**
- probability X is below its qth quantile is q. In other words,

$$P\{X \leq F^{-1}(q)\} = q$$

- **median** is the 50% percentile or .5 quantile
- $\sigma_{XY} = E\Big[\{X - E(X)\}\{Y - E(Y)\}\Big]$
- **correlation coefficient** between X and Y:

$$\rho_{XY} := \sigma_{XY}/\sigma_X\sigma_Y$$

- the best linear predictor of Y is

$$\widehat{Y} = E(Y) + \frac{\sigma_{XY}}{\sigma_X^2}\{X - E(X)\}$$

- expected squared prediction error is

$$E\{Y - \widehat{Y}\}^2 = \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2} = \sigma_Y^2(1 - \rho_{XY}^2)$$

- if X is not observed then $\widehat{Y} = E(Y)$

- expected squared prediction error is σ_Y^2

- if $X \sim N(\mu, \sigma^2)$ then $P(X \leq x) = \Phi\{(x-\mu)/\sigma\}$.

$$E(aY + b) = aE(Y) + b$$

$$\text{Var}(aY + b) = a^2\text{Var}(Y).$$

$$E(w_1X + w_2Y) = w_1E(X) + w_2E(Y),$$

$$\text{Var}(w_1X + w_2Y) = w_1^2\text{Var}(X) + 2w_1w_2\text{Cov}(X, Y) + w_2^2\text{Var}(Y)$$

- Suppose

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}$$

Then, the expectation of $\boldsymbol{X}$ is

$$E(\boldsymbol{X}) := \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_N) \end{pmatrix}.$$

and the covariance matrix of $\boldsymbol{X}$ is

$$\text{COV}(\boldsymbol{X}) := \begin{pmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_N) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) & \cdots & \text{Cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(X_N, x_1) & \text{Cov}(X_N, X_2) & \cdots & \text{Var}(X_N) \end{pmatrix}.$$

- The *p-value* for a sample is defined as the smallest value of α for which the null hypothesis is rejected for that sample.

- $L(\boldsymbol{\theta}) := f(\boldsymbol{Y}; \boldsymbol{\theta})$ is the "likelihood function"

- maximum likelihood estimator = MLE = value of $\boldsymbol{\theta}$ that maximizes $L(\boldsymbol{\theta})$

- For some models such as the ARIMA time series models, there are software packages, e.g, MINITAB and SAS, that compute the MLE

- The likelihood ratio test rejects H_0 if

$$\begin{aligned} & 2\log\left\{\frac{L(\widehat{\boldsymbol{\theta}}_{ML})}{L(\boldsymbol{\theta}_{1,0},\widehat{\boldsymbol{\theta}}_{2,0})}\right\} \\ = \;& 2\Big[\log\{L(\widehat{\boldsymbol{\theta}}_{ML})\} - \log\{L(\boldsymbol{\theta}_{1,0},\widehat{\boldsymbol{\theta}}_{2,0})\}\Big] \geq \chi^2_{\alpha;\,\dim(\boldsymbol{\theta}_1)} \end{aligned}$$

  - $\dim(\boldsymbol{\theta}_1)$ = number of components of $\boldsymbol{\theta}_1$
  - $\chi^2_{\alpha,k}$ is the α upper-probability value of the chi-squared distribution with k degrees of freedom

Chapter 3

Returns: 3/24/02

3.1 Prices and returns

Let P_t be the price of an asset at time t.

Assuming no dividends the *net return* is

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}$$

The *simple gross return* is

$$\frac{P_t}{P_{t-1}} = 1 + R_t$$

Example: If $P_t = 2$ and $P_{t+1} = 2.1$ then

$$1 + R_t = 1.05 \quad \text{and} \quad R_t = .05.$$

The gross return over the most recent k periods ($t - k$ to t) is

$$\begin{aligned} 1 + R_t(k) &:= \\ \frac{P_t}{P_{t-k}} &= \left(\frac{P_t}{P_{t-1}}\right)\left(\frac{P_{t-1}}{P_{t-2}}\right)\cdots\left(\frac{P_{t-k+1}}{P_{t-k}}\right) \\ &= (1 + R_t)\cdots(1 + R_{t-k+1}) \end{aligned}$$

Returns are scale-free, meaning that they do not depend on units (dollars, cents, etc.). Returns are *not* unitless. Their unit is time; they depend on the units of t (hour, day, etc.).

Example:

Time	$t-2$	$t-1$	t	$t+1$
P	200	210	206	212
$1+R$		1.05	.981	1.03
$1+R(2)$			1.03	1.01
$1+R(3)$				1.06

3.2 Log returns

Continuously compounded returns, also known as "log returns" are:

$$r_t := \log(1+R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = p_t - p_{t-1}$$

where

$$p_t := \log(P_t)$$

[**Notation:** $\log(x)$ will mean the natural logarithm of x throughout these notes. $\log_{10}(x)$ will be used to denote the logarithm to base ten, if it is needed.]

Log returns are approximately equal to returns:

- x small, $\log(1+x) \approx x$
- **Examples:**
  - $\log(1+.05) = .0488$
  - $\log(1-.05) = -.0513$
- see Figure 3.1
- therefore, $r_t = \log(1+R_t) \approx R_t$.

Advantage — simplicity of multiperiod returns

$$\begin{aligned} r_t(k) &:= \log\{1+R_t(k)\} \\ &= \log\{(1+R_t)\cdots(1+R_{t-k+1})\} \\ &= \log(1+R_t)+\cdots+\log(1+R_{t-k+1}) \\ &= r_t + r_{t-1} + \cdots + r_{t-k+1} \end{aligned}$$

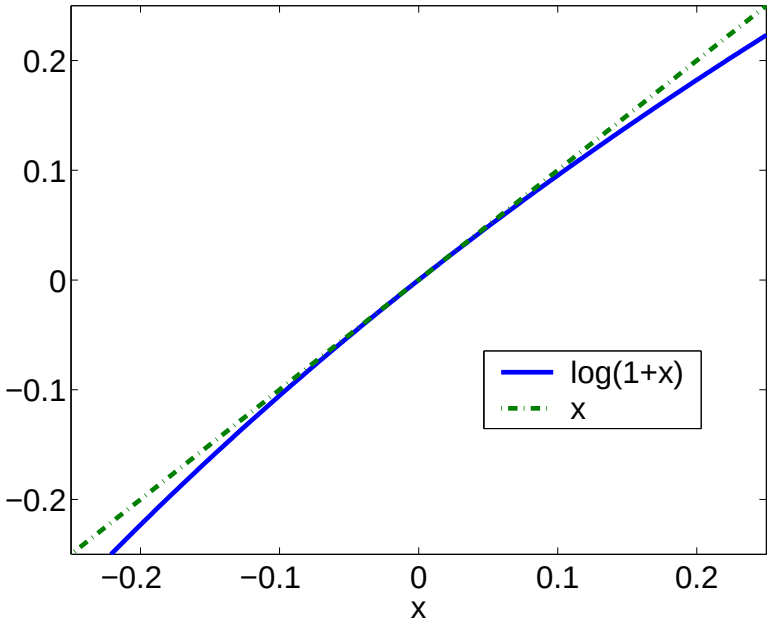


Figure 3.1: *Comparison of functions* $\log(1+x)$ *and* x.

3.3 Behavior of returns

What can we say about returns?

- They cannot be perfectly predicted — i.e., they are random.
- If we were ancient Greeks, we would think of the returns as determined by the Gods or Fates (three Goddesses of destiny). The Greeks did not seem to realize that random phenomena do exhibit some regularities such as the law of large numbers and the central limit theorem.

Peter Bernstein has written an interesting popular book "Against the Gods: The Remarkable Story of Risk." He chronicles the developments of probability theory and our understanding of risk.

It took a surprisely long time for probability theory to develop. The ancient Greeks did not have probability theory.

Probability arose out of gambling during the Renaissance.

University of Chicago economist Frank Knight (1916 Cornell PhD) distinguishes between

- measurable uncertainty or "risk proper" (e.g., games of chance) where the probabilities are known
- unmeasurable uncertainty (e.g., finance) where the probabilities are unknown

At time $t-1$ P_t and R_t are not only unknown, but we do not know their probability distributions.

However, we can estimate these probability distributions *if* we are willing to make an assumption.

Leap of Faith

Future returns will be similar to past returns.

More precisely, the probability distribution of P_t can be determined (or at least estimated) from past data.

With this (big) assumption, we can get somewhere — and we will!

Asset pricing models (e.g. CAPM) use the joint distribution of a cross-section $\{R_{1t}, \ldots, R_{Nt}\}$ of returns on N assets at a single time t. Here R_{it} is the return on the ith asset at time t.

Other models use the time series $\{R_1, R_2, \ldots, R_t\}$ of returns on a single asset at times $1, 2, \ldots, t$. We will start with a single asset.

3.4 Common Model — IID Normal Returns

Here $R_1, R_2, \ldots$ are the returns from a single asset. A common model is that they are

1. mutually independent
2. identically distributed, i.e., they have the probability distributions and in particular same mean and variance.
3. normally distributed

IID = **i**ndependent and **i**dentically **d**istributed

There are (at least) two problems with this model:

- The model implies the possibility of unlimited losses, but liability is usually limited; $R_t \geq -1$ since you can lose no more than your investment
- $1 + R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})$ is not normal — sums of normals are normal but not so with products.

3.5 The Lognormal Model

A second model assumes that the continuously compounded single-period returns, also known as the log returns and denoted by r_t, are IID. Recall that the log return is

$$r_t = \log(1 + R_t)$$

where $1 + R_t$ is the simple gross return

Thus, we assume that

$$\log(1 + R_t) \sim N(\mu, \sigma^2)$$

so that $1 + R_t = \exp(\text{normal r.v.}) \geq 0$ so that $R_t \geq -1$. This solves the first problem.

Also,

$$\begin{aligned} 1 + R_t(k) &= (1 + R_t) \cdots (1 + R_{t-k+1}) \\ &= \exp(r_t) \cdots \exp(r_{t-k+1}) \\ &= \exp(r_t + \cdots + r_{t-k+1}). \end{aligned}$$

Therefore,

$$\log\{1 + R_t(k)\} = r_t + \cdots r_{t-k+1}$$

Sums of normals are normal $\Rightarrow$ the second problem is solved — normality of single period returns implies normality of multiple period returns.

The lognormal distribution goes back to Louis Bachelier (1900).

- dissertation at Sorbonne called *The Theory of Speculation*
- Poincarè: "M. Bachelier has evidenced an original and precise mind [but] the subject is somewhat remote from those our other candidates are in the habit of treating."
- Bachelier was awarded "mention honorable" rather than "mention trés honorable" — Bachelier never found a decent academic job.
- Bachelier anticipated Einstein's (1905) theory of Brownian motion.

In 1827, Brown, a Scottish botanist, observed the erratic, unpredictable motion of pollen grains under a microscope.

Einstein (1905) — movement due to bombardment by water molecules — Einstein developed a mathemetical theory giving precise quantitative predictions.

Later, Norbert Wiener, an MIT mathematician, developed a more precise mathematical model of Brownian motion. This model is now called the Wiener process.

[Aside: 1905 was a good year for Einstein. He published:

- the paper introducing the theory of special relativity
- a paper on quantization of light which led quantum theory (which he never embraced — "God does not play dice with the world")
- the paper on Brownian motion]

Bachelier stated that

- "The mathematical expectation of the speculator is zero" (this is essentially true of short-term speculation but not of long term investing, since the stock market does go up on average)
- "It is evident that the present theory solves the majority of problems in the study of speculation by the calculus of probability"

Bachelier's thesis came to light accidently more than 50 years after he wrote it. The statistician and economist Jimmie Savage found a book by Bachelier in the U. Chicago library and asked other economists about it. Paul Samuelson found Bachelier's thesis in the MIT library. The English translation was published in 1964 in *The Random Character of Stock Market Prices,* an edited volume.

Example: A simple gross return, $(1+R)$, is lognormal(0,$(.1)^2$), which means that $\log(1+R)$ is $N(0,(.1)^2)$. What is $P(1+R<.9)$?

Answer: Since log(.9)$=-.105$,

$$P(1+R<.9)=P\{\log(1+R)<\log(.9)\}=\Phi\{(-.105-0/.1\}=\Phi(-1.05)=.1469.$$

In MATLAB, `cdfn(-1.05) = .1469`.

Example:

Assume again that $1+R$ is lognormal(0,$(.1)^2$). Find the probability that a simple gross two-period return is less than .9.

Answer: The two-period gross return is log normal$(0,2(.1)^2)$ so this probability is

$$\Phi\left\{\frac{\log(.9)}{(\sqrt{2})(.1)}\right\}=\Phi(-.745)=.2281.$$

Let's find a general formula for the kth period returns:

Assume that

- $1 + R_t(k) = (1 + R_t) \cdots (1 + R_{t-k+1})$.
- $\log(1 + R_i) \sim N(\mu, \sigma^2)$ for all i
- The $\{R_i\}$ are mutually independent.

Then $\log\{1 + R_t(k)\}$ is the sum of k independent $N(\mu, \sigma^2)$ random variables, so that $\log(1 + R_t(k)) \sim N(k\mu, k\sigma^2)$.

$$P\{1 + R_t(k) < x\} = \Phi\left\{\frac{\log(x) - k\mu}{\sqrt{k}\sigma}\right\}.$$

3.6 Random Walk

Let $Z_1, Z_2, \ldots$ be IID with mean μ and standard deviation σ. Z_0 is an arbitrary starting point. Let $S_0 = Z_0$ and

$$S_t := Z_0 + Z_1 + \cdots + Z_t, \ t \geq 1.$$

$S_0, S_1, \ldots$ is called a random walk. We have $E(S_t|Z_0) = Z_0 + \mu t$ and $\text{Var}(S_t|Z_0) = \sigma^2 t$. The parameter μ is called the "drift" of the random walk and it determines the general direction of the random walk. The parameter σ is the volatility and determines how much the random walks fluctuates about the mean $Z_0 + \mu t$.

3.6.1 Geometric Random Walk

Recall that $\log\{1 + R_t(k)\} = r_t + \cdots + r_{t-k+1}$. Therefore

$$\frac{P_t}{P_{t-k}} = 1 + R_t(k) = \exp(r_t + \cdots + r_{t-k+1})$$

so taking $k = t$ we have

$$P_t = P_0 \exp(r_t + r_{t-1} + \cdots + r_1).$$

Conclusion: If the log returns are IID normals, then the process $\{P_t : t = 1, 2, \ldots\}$ is the exponential of a random walk. We call such a process a "geometric random walk" or an "exponential random walk."

If $r = \log(1 + R)$ is $N(\mu, \sigma^2)$, then the median of R is $\exp(\mu) - 1$ since

$$P(R < \exp(\mu) - 1) = P(1 + R < \exp(\mu)) = P(r < \mu) = P(N(\mu, \sigma^2) < \mu) = \frac{1}{2}.$$

$E(R) = \exp(\mu + \sigma^2/2) - 1 >$ median

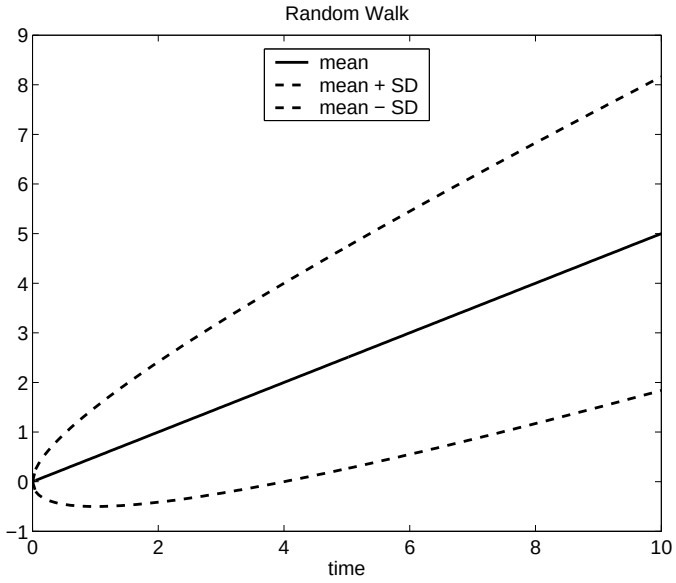


Figure 3.2: *Mean and probability bounds on a random walk with* $S_0 = 0$, $\mu = .5$ *and* $\sigma = 1$. *At any given time, the probability of being between the probability bounds (dashed curves) is 68%.*

3.6.2 The effect of the drift μ

The geometric random walk model does *not* imply that one cannot make money in the stock market. Quite the contrary, since μ is generally positive, there is an upward "drift" to the random walk. For example, the log returns on the US stock market as a whole have a mean of about 10% and a standard deviation of about 20%. Let's look at 20 years of geometric random walk with $\mu = .1$ and $\sigma = .2$. The expected log return on each step is .1. The expected log return for 20 years would be (20)(.1) = 2.0. The median gross return is $\exp(2) = 7.38$. If the stock price starts at \$100, the median price after 20 years is \$738.[1] The expected price after 20 years is $\$100\exp\{(20)(.1) + (20)(.2)^2/2\} = 1,102$.[2]

Figure 3.4 shows two independent simulations of 20 years of a geometric random walk with these values of μ and σ. The prices start at 100. Notice that the two series of log returns look quite different. However, the

[1] The general formula for the median price after k years is $P_0 \exp\{k\mu\}$ where P_0 is the starting price.

[2] The general formula for the expected price after k years is $P_0 \exp\{k\mu + k\sigma^2/2\}$.

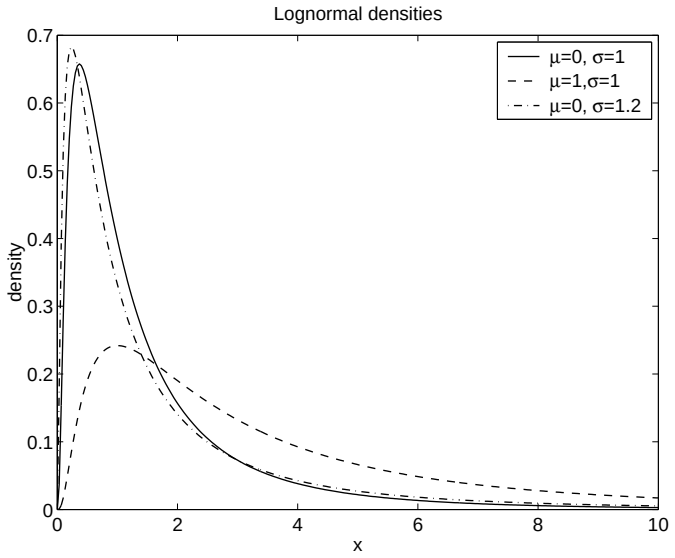


Figure 3.3: *Log normal densities.*

log prices and prices looks much more similar. The reason is that they both have $\mu = .1$ which over the long run gives an upward trend. In the plots, the medians of the log returns, log prices, and prices are also plotted. They are smooth curves.

3.7 Are log returns really normally distributed?

There are several ways to check whether log returns are really normally distributed. One way is to look at a normal probability plot of the log returns to see if the plot is approximately a straight line. Another method is to look at the sample skewness and kurtosis of the log returns and to check if their values are near those of the normal distribution. Skewness and kurtosis measure the "shape" of a probability distribution and are independent of the mean and variance. Everu normal distribution has a skewness coefficient of 0 and a kurtosis of 3.

Suppose with have a time series of log returns, $r_1, \ldots, r_t, \ldots, r_T$ on some asset. The sample mean and standard deviation of this series are

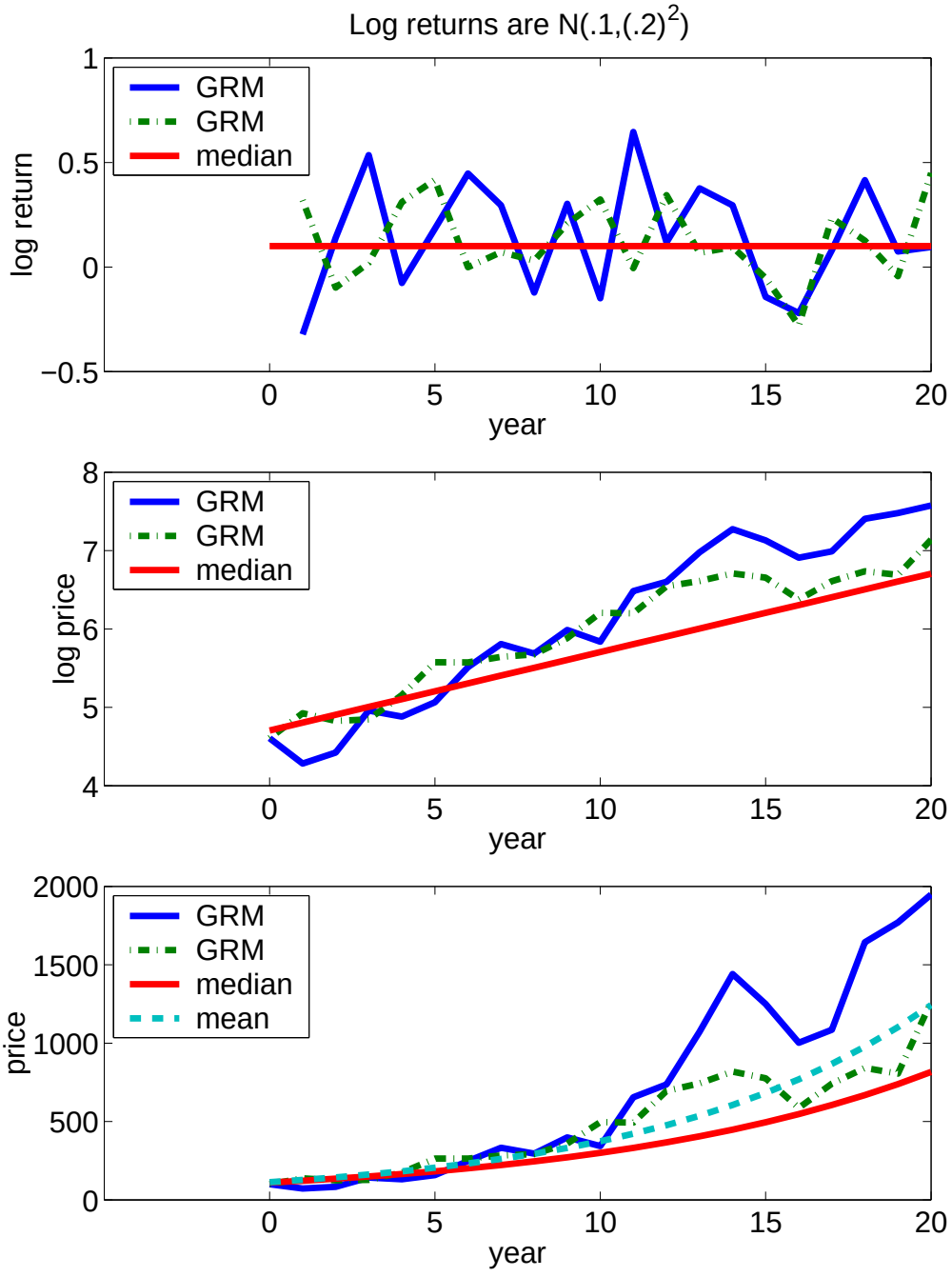


Figure 3.4: *Two independent simulations of a geometric random walk with* $\mu = .1$ *and* $\sigma = .2$. *Medians are shown as solid curves. Means are shown as dashed curved in the case of the lognormally distributed prices where the mean does not equal the mediam.*

$\widehat{\mu}$ and $\widehat{\sigma}$. The sample skewness, denoted by $\widehat{S}$, is

$$\widehat{S} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_t - \widehat{\mu}}{\widehat{\sigma}}\right)^3.$$

The sample kurtosis, denoted by $\widehat{K}$, is

$$\widehat{K} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_t - \widehat{\mu}}{\widehat{\sigma}}\right)^4.$$

The "excess kurtosis" is $\widehat{K} - 3$ and measures deviation from 3, the kurtosis of a normal distribution. Both the sample skewness and the excess kurtosis should be near 0 if the log returns are normally distributed.

Table 1.1 of Campbell et al. gives $\widehat{S}$ and $\widehat{K} - 3$ for several market indices and common stocks. In that table, $\widehat{S}$ is generally close to zero, which indicates that log returns are not very skewed. However, the excess kurtosis is typically rather large for daily returns and positive though not as large for monthly returns. By the CLT, the distribution of log returns over longer periods should approach the normal distribution. Therefore, the smaller excess kurtosis for monthly log returns, in contrast to daily log returns, is expected. The large kurtosis of daily returns indicates that they are "heavy-tailed."

Normal probability plots can be supplemented by tests of normality based on the sample CDF, $\widehat{F}$. $\widehat{F}(x)$ is defined to be the proportion of the sample that is less than or equal to x; for example if 10 out of 40 data points are 3 or less then $\widehat{F}(3) = .25$. Normality is tested by comparing the sample CDf with the normal CDF with mean and variance equal to the sample mean and variance, i.e., with compare $\widehat{F}(x)$ with $\Phi\{(x - \overline{x})/s\}$. Three common tests of normality that compare the sample CDF with the normal CDF are the Anderson-Darling test, the Shapiro-Wilks test, and the Kolmogorov-Smirnov test. All three are available on MINITAB. Actually, MINITAB uses the Ryan-Joiner test which is close to the Shapiro-Wilks test. In MINITAB, go to "Stat," then "Basic Statistics," and then "Normality test." You will need to choose one the three tests. The output is a normal plot plus the results of the test. You can re-run the procedure to perform the other tests.

The Kolmogorov-Smirnov test is based on the maximum distance between the sample CDF and the normal CDF.

The Shapiro-Wilks test is closely tied to the normal probability plot, since it is based on the correlation between the normal quantiles and the

sample quantiles. The correlation measures how close the normal plot is to being a straight line.

CDF's and quantiles are closely related. In fact, as discussed in Section 2.1, quantiles are given by the inverse of the CDF function; if a random variable X has CDF F then the pth quantile of X is $F^{-1}(p)$ since $P\{X \leq F^{-1}(p)\} = F\{F^{-1}(p)\} = p$.

Let's look at daily returns for GE common stock from December 1999 to December 2000. The daily price P_t is taken to be the average of the high and the low for the day. It might have been better to use the closing price for each day. Why?

As can be seen in Figure 3.5, the net returns R and the log returns r are *very* similar. The normal plot shown in the second row, second column is roughly linear.

The log return have a sample mean, standard deviation, skewness, and excess kurtosis of .00014, .0176, $-.094$, and .094, respectively. The values of the sample skewness and excess kurtosis are close to zero and suggest that the log returns are approximately normally distributed.

From MINITAB, the Kolmogorov-Smirnov, Anderson-Darling, and Ryan-Joiner tests of normality have a p-values of .15, .40, and .10, respectively. Since each p-value exceeds .1, each test would accept the null hypothesis of normality at $\alpha = .1$ and of course at $\alpha = .05$.

3.7.1 Do the GE daily returns look like a geometric random walk?

Figure 3.6 shows five independent simulated geometric random walks with the same parameters as the GE daily log returns. Note that the geometric random walks seem to have "patterns" and "momentum" even though they do not. The GE log returns look similar to the geometric random walks.

It is somewhat difficult to distinguish between a random walk and a geometric random walk. Figure 3.7 shows three independent simulated time series. For each pair, the log price series (a random walk) is plotted on the left while the price series (a geometric random walk) is plotted on the right. Note the subtle differences between the prices and the log prices.

We prefer the geometric random walk model to the random walk model, because the geometric random walk model is more realistic: the geometric random walk implies non-negative prices and net returns that are at least -1.

This graphical comparison of GE prices to geometric random walks is *not* strong evidence in favor of the geometric random walk hypothesis. This

hypothesis implies that the log returns are *mutually independent* and, therefore, uncorrelated. Therefore we should check for evidence that the log returns are correlated. If we find no such evidence, then we have more reason to believe the geometric random walk hypothesis. We will return to this issue in Chapter 4 after we have studied ways of detecting correlation.

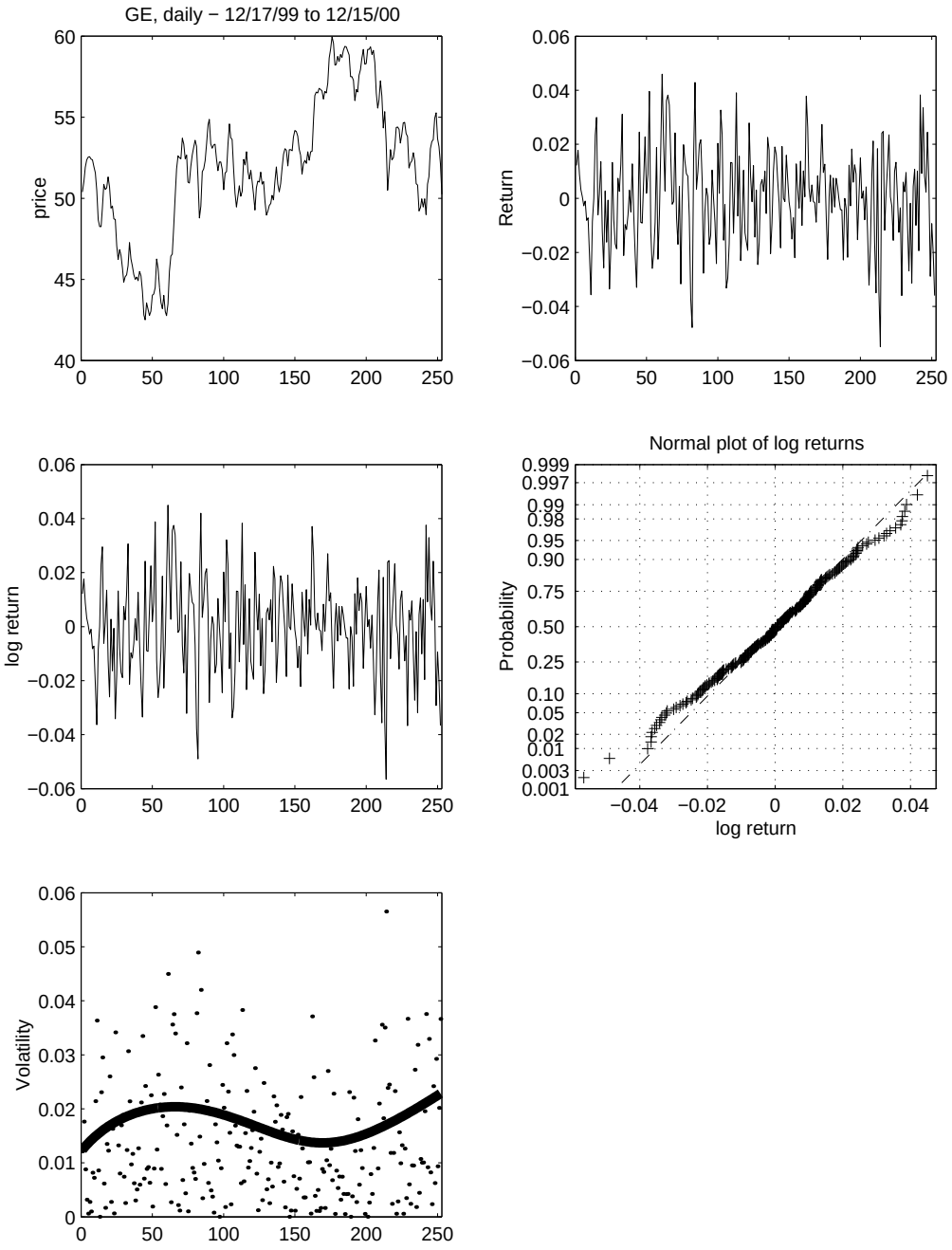


Figure 3.5: *GE daily returns. The first plot is the prices. The second and third are the net returns and the log returns. The fourth plot is a normal probability plot of the log returns. The final plot is of the absolute log returns; there is a scatterplot smooth to help show whether the volatility is constant.*

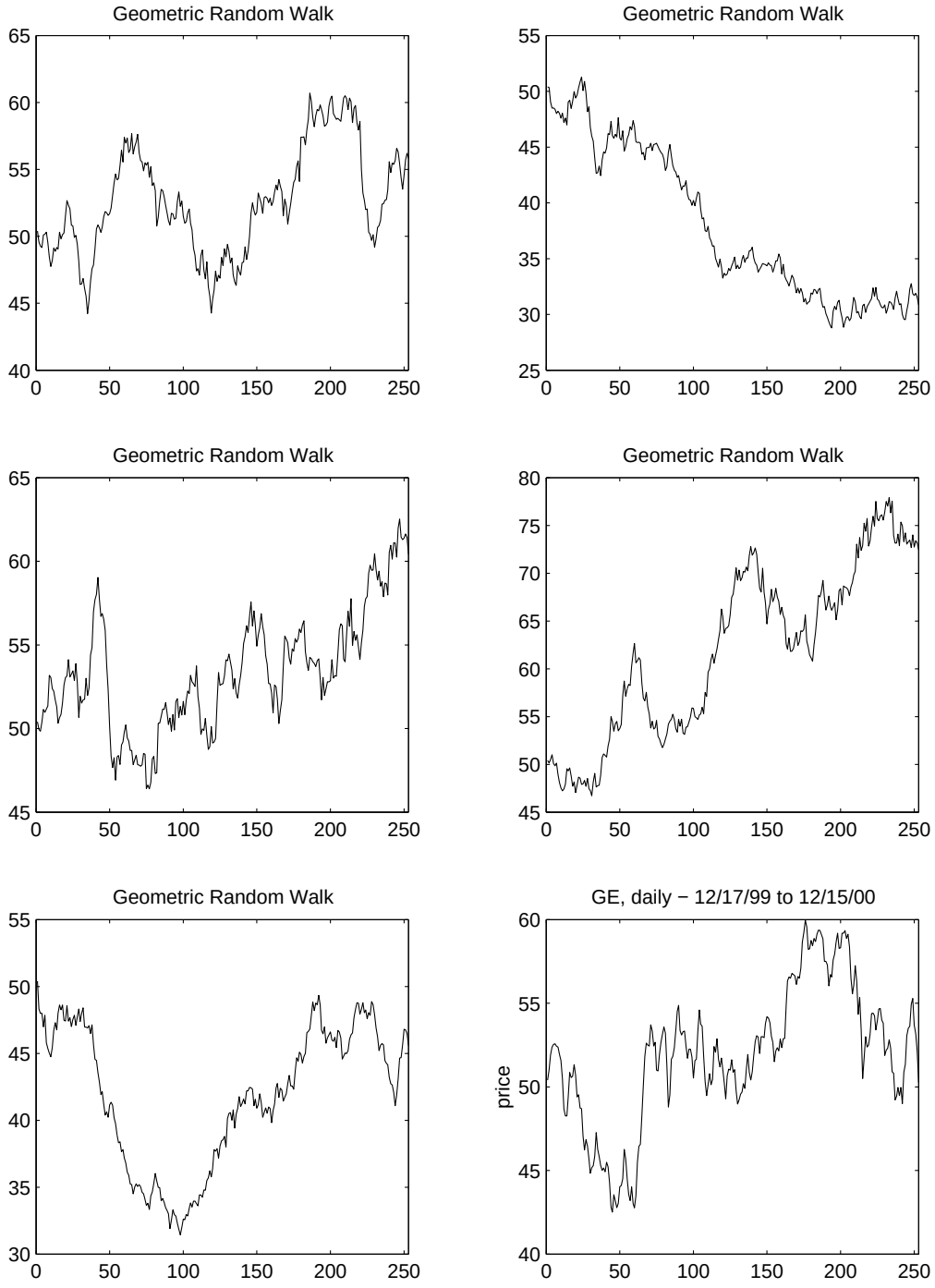


Figure 3.6: *Five independent geometric random walks and GE daily log returns. The geometric random walks have the same expected log return, volatility, and starting point as the GE log returns.*

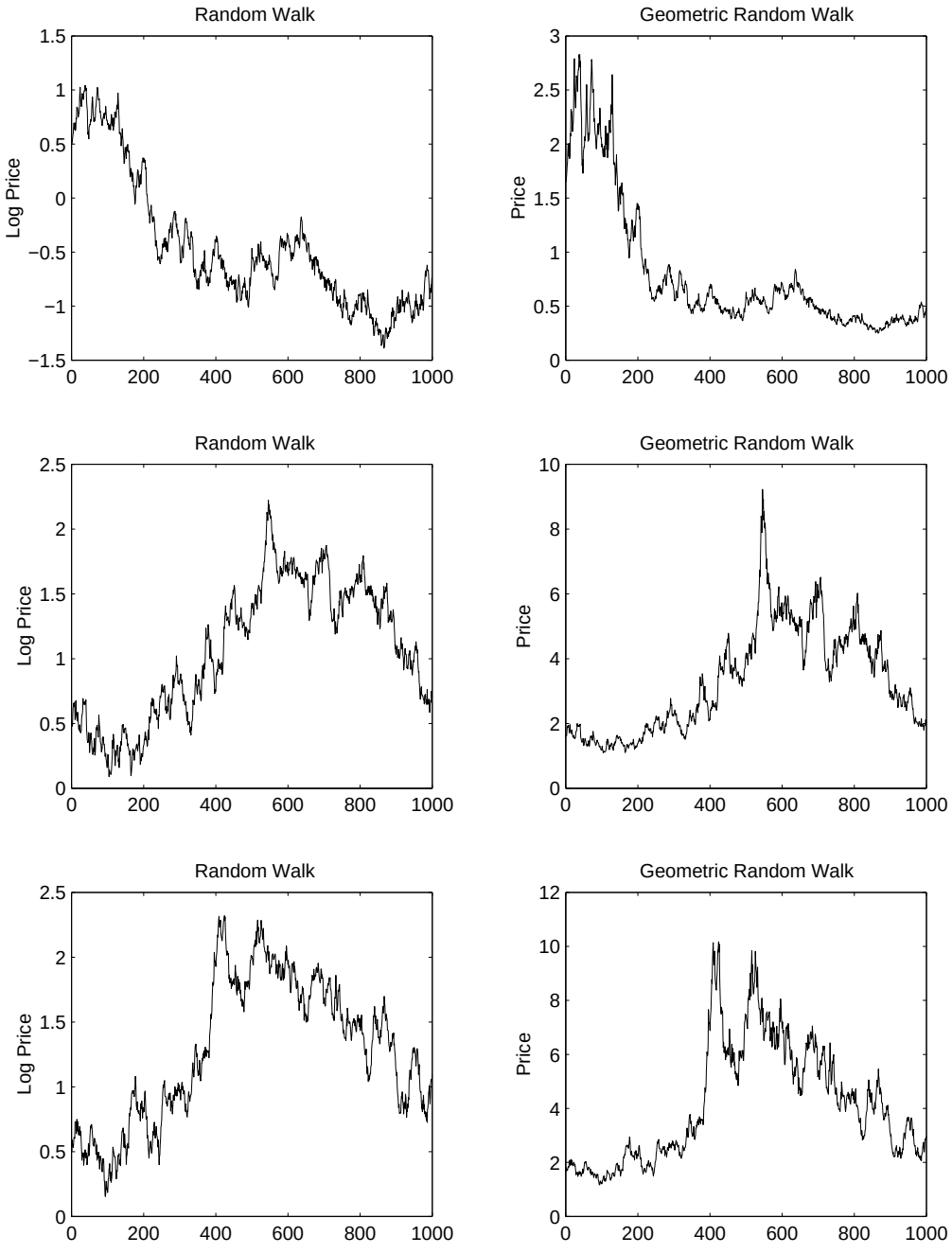


Figure 3.7: *Three independent simulated price series. On left: log prices. On right: prices.*

3.8 Portrait of an econometrician, Eugene Fama

This material is taken from Chapter 7 of *Capital Ideas* by Peter Bernstein.

Fama was born in 1939 in Boston, majored in French at Tufts, and was an outstanding student-athlete.

In college, Fama earned extra money working for Harry Ernst who published a stock market newsletter:

- Fama's job was to find workable buy and sell signals.
- Ernst believed that trends, once in place, would continue because of "price momentum."
- Bernstein writes that "Fama's efforts to develop profitable trading rules were by no means unsuccessful" but "the ones he found worked only on the old data, not on the new."
  - like many other investors Fama found that rules that worked well on "backtests" couldn't beat the market when applied in real time.
  - the market environment would shift or too many people would be using the same strategy

Fama decided to go to business school to learn what was really going on.

- 1964 doctorate at University of Chicago.
- he thought of going to Harvard but was told that he was "more intellectual than the typical Harvard type"

Fama stayed at Chicago where he taught finance.

- scholars at Chicago were keenly interested in collecting facts (empirical research!)
- at Chicago, James Lorie and Lawrence Fisher were demonstrating what the computer could offer to economic research
  - 1964: Lorie and Fisher published a "bombshell" — \$1000 invested in 1926 would grow to almost \$30,000 in 1960, a growth of over 9% a year (log(30/35) = .097)

    - Remember: 1929 was the great crash and the ensuing great depression lasted until the US entered WW II in the 40's. This was not exactly a favorable time for investing.
    - These findings increased the interest in stocks as long-term investments.

- 1965: Fama published "The Behavior of Stock Market Prices" (his thesis) in *Journal of Business*.
  - a less technical version was published in 1966 as "Random Walks in Stock Market Prices" in *Financial Analysts Journal.*
    - the less technical version was reprinted in *Institutional Investor*.

- Fama's first target was "technical analysis" as practiced by so-called "chartists."
  - technical analysts believe that future prices can be predicted from past patterns
  - Charting stock prices was once fashionable
    - I remember as a young child my grandmother explaining to me how to chart stock prices.
  - Fama: "The chartist must admit that the evidence in favor of the random walk model is both consistent and voluminous, whereas there is precious little published in discussion of rigorous empirical tests of various technical theories."

- Fama's next target was "fundamental analysis" as practiced by securities analysts.
  - Fundamental analysts examine accounting data, interview management, and look at economic forecasts, interest rates, and political trends.
  - Selecting stocks by fundamental analysis seems to do no better than using a dartboard.
  - Of course, good management, favorable economic trends, etc. influence the prices of assets, but Fama claimed that this information is already fully reflected in stock prices by the time we learn it — markets react instantaneously to information.

  - Security analysis is essential in order for stocks to be priced correctly, but ironically it means that there are few discrepancies between actual prices and the values of stocks
  - William Sharpe discussed the antagonism of professional investors to the random walk theories of Fama and other academics. He stated that "Interestingly, professional economists seem to think more highly of professional investors than do other professional investors."
    * What Sharpe is saying here is that professional economists believe that professional investors are so good that stocks are properly priced, while professional investors think that other professional investors as so incompetent that there are bargains out there waiting to be purchased by a few smart investors like themselves.
    * Later we will learn more about Sharpe, the economist who developed the CAPM and winner of the Nobel Prize.

3.9 Other empirical work related to Fama's

Fama's work was preceded by that of other researchers.

- In 1933 Alfred Cowles published "Can stock market forecasters forecast?" The three-word abstract stated "It is doubtful." The article appeared in the brand-new journal *Econometrica*. *Econometrica* is now the leading journal in econometrics, which is the empirical modeling and analysis of economic data[3].

  - Cowles analyzed the track records of:
    * 16 leading financial services that furnished their subscribers with selected lists of common stocks
    * purchases and sales of stock by 20 leading fire insurance companies
    * 24 publications by financial services, financial weeklies, bank letters, etc.
    * editorials in *The Wall Street Journal* by William Peter Hamilton, an expounder of the "Dow Theory" due to Charles Dow (the Dow of Dow-Jones). Dow compared stock prices to

[3]This course could be called the econometrics of financial data.

tides and ocean waves; the tides were a metaphor to explain "price momentum."

- Cowles found that only 6 of 16 financial services had achieved any measure of success
  - even the best record could not be definitely attributed to skill rather than luck
    - one needs statistical analysis to reach such a conclusion
    - the null hypothesis to test is that the record of an investor is no better than would be expected by chance
- In 1944, Cowles published a new study with basically the same conclusions.

- In 1936, Holbrook Working published a paper in *The Journal of the American Statistical Association* on commodity prices.
  - These were once believed to have rhythms and trends.
  - Working found that he could not distinguish the price *changes* from an independent sequence of random changes.
  - Perturbed, Working took his data to professional commodity traders.
    - He also showed them graphs of random series.
    - The professionals could not distinguish the random series from real commodity prices.
    - Of course, Working's study of commodity prices does not prove anything about stock returns, but it is an interesting example of a financial time series where momentum was thought to exist, but where no evidence of momentum was found in a statistical analysis.

- Maurice Kendall published the paper "The analysis of economic time series" in the *Journal of the Royal Statistical Society* in 1953.
  - Kendall wrote "the patterns of events in the price series was much less systematic than is generally believed," and
  - "Investors can, perhaps, make money on the Stock Exchange, but not, apparently by watching price movements and coming in on what looks like a good thing ... But it is unlikely that anything I say or demonstrate will destroy the illusion that the outside investor can make money by playing the markets, so let us leave him to his own devices."

- There is no question as to whether one can make money in the stock market.
  - Over the long haul, stocks outperform bonds which outperform savings accounts.
  - The question is rather whether anyone can "beat the market."
    - "Beating the market" means an individual's portfolio outperforming a market index such as the Dow Jones average or the NASDAQ index.
    - The outperformance should not be explanable as reasonably due to chance.
    - There should be some correction for the degree of risk the individual has assumed since risk premiums do exist.
      - if an investor picks risky stocks, then his expected return is higher
      - the question is whether his expected return is higher than that of a market index with the same level of risk.

3.10 Technical Analysis

A Random Walk Down Wall Street was written by Burton G. Malkiel, a professor of economics at Princeton. It is a perennial best seller and has been revised several times. It contains much sensible advice for the small investor. This book is also quite humorous, and the discussion of technical analysts is particularly amusing (unless you are a technical analyst).

Malkiel writes

> I, personally, have never known a successful technician, but I have seen the wrecks of several unsuccessful ones. (This is, of course, in terms of following their own technical advice. Commissions from urging customers to act on their recommendations are very lucrative.)

Malkiel describes many of the technical theories, including the Dow Theory, the Filter System, and the Relative-Strength system. The latter advises buying stocks that have done well recently. There is also the hemline theory which predicts price changes by the lengths of women's dresses and the super bowl indicator which says that "a victory by an NFL team predicts a bull market, whereas a victory by a former AFL team is bad news for stock-market investors."

There is also the odd-lot theory. It is based on the impeccable logic that a person who is *always* wrong is a reliable source of information—just negate whatever that person says. The assumption of the odd-lot theory is that the odd-lot trader is precisely that sort of idiot. It turns out that the odd-lotter isn't such a dolt after all. Research suggest that odd-lotter do not do worse than average. Yes, we can make a ton of money if we can find an investor who is always wrong, but finding such an investor is just as difficult as finding one who is always correct. If stocks are correctly priced as empirical research suggests, then not only is it impossible to beat the market, but it is impossible to do worse than the market[4].

Human nature seems to find randomness very hard to accept. For example, sports fans have many theories of streaks in athletics, e.g., the "hot hand" theory of basketball. Extensive testing of basketball players' performances have show no evidence of streaks beyond what would be expected by pure chance. The point is that streaks will occur by chance, but you cannot make money on the basis of random streaks since you cannot predict if they will continue.

Why are technicians hired? Malkiel has the skeptical view that it is because their theories recommend a lot of trading. "The technicians do not help produce yachts for the customers, but they do help generate the trading that provides yachts for the brokers."

3.11 Fundamental Analysis

The practitioners of fundamental analysis are called *security analysts*. Their job is basically to predict future earnings of companies, since it is future earnings that ultimately drive prices.

Although few on Wall Street still have much faith in technical analysis, there is still much faith in fundamental analysis. However, some academics studying the financial markets data have come to the conclusion that security analysts can do no better than blindfolded monkeys who throw darts at the *Wall Street Journal*.

[4]Although stupidity cannot get you a smaller expected return, it can get you a larger risk than necessary if you fail to diversify as discussed in Chapters 5 and 7.

3.12 Efficient Markets Hypothesis (EMH)

As evidence accumulated that stock price fluctuated like random walks, economists sought a theory as to why that would be so. In 1965 Paul Samuelson published a paper "Proof that properly anticipated prices fluctuate randomly." The idea is that random walk behavior is due to the very efficiency of the market.

- A market is information efficient if prices "fully reflect" available information
- A market is "efficient with respect to an information set" if prices would be unchanged by revealing that information to *all* participants
  - this implies that it is impossible to make economic profits by trading on the basis of this information set
- This last idea is the key to testing (empirically) the EMH.

Samuelson's important idea is that under an efficient market prices will chance only when there is new and unanticipated information. Since the information, and therefore the price changes, are unanticipated, price changes will be random. The market does not react to events that are expected. For example, stock prices dropped instantaneously when markets reopened after the September 11th surprise attacks. A few weeks later when the US started to bomb the Taliban there was little or no market reaction because investors already knew that the bombing was coming.

3.12.1 Three types of efficiency

weak-form efficiency the information set includes only the history of prices or returns

semi-strong efficiency the information set includes all information that is publically available

strong-form efficiency the information set includes all information known to any market participant

Weak-form efficiency $\Rightarrow$ technical analysis will not make money

Semistrong-form efficiency $\Rightarrow$ fundamental analysis will not help the average investor

3.12.2 Testing market efficiency

The research of Fama, Cowles, Working, and Kendall just described tests the various forms of the EMH. Cowles's work supports the semi-strong and perhaps the strong form of the EMH.

In their book *Investments*, Bodie, Kane, and Marcus discuss some of the issues involved when testing the EMH. One is the magnitude issue. No one believes that markets are perfectly efficient. The small inefficiencies might be important to the manager of a large portfolio. If one is managing a $5 billion portfolio, beating the market by .1% results in a $5 million increase in profit. This is clearly worth achieving. Yet, no statistical test is likely to undercover a .1% inefficiency amidst typical market fluctuations, since the latter are large. For example, the S&P 500 index has a 20% standard deviation in annual returns.

Another issue is selection bias. If there is someone who can consistently beat the market, they probably are keeping that a secret. We can only test market efficiency by testing methods of technical or fundamental analysis that are publicized. These may be the ones that don't reveal market inefficiencies.

Another problem is that for any time periods, *by chance* there will be some investment managers who consistently beat the market.

- if 2,000 people each toss a coin 10 times, it is likely that at least one will get 10 heads since the expected number of people tossing 10 heads is[5]

  $$2,000 * 2^{-10} = 1.95.$$

  Using the Poisson approximation to the binomial, the probability that no one tosses 10 heads is $\exp(-1.95) = .1423$. The exact probability[6] is $(1023/1024)^{2000} = .1417$.

  If someone does toss 10 heads, it would be a mistake to say that this person has skill in tossing heads.

- Peter Lynch's Magellan Fund outperformed the S&P 500 in 11 of 13 years ending in 1989. Was Lynch a skilled investment manager or just lucky? (If he really was skilled, then this is evidence against the semi-strong form of the EMH.)

[5] This is the binomial expectation np with $n = 2000$ and $p = 2^{-10}$.

[6] This is the binomial probability $\binom{n}{x} p^x (1-p)^{n-x}$ with $n = 2000$, $x = 0$, and $p = 1/1024$.

Campbell, Lo, and MacKinlay and Bodie, Kane, and Marcus discuss much of the empirical literature on testing the EMH and give references to the original studies.

Fama was written a review article:

Fama, E. (1970), "Efficent Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance*, 25, 383–417.

There is a sequel as well:

Fama, E., (1991), "Efficient Capital Markets: II," *Journal of Finance*, 46, 1575–1618.

The *Journal of Finance*, as well as many other journals in economics and finance, are available online at JSTOR:

http://www.jstor.org/cgi-bin/jstor/listjournal

However, the most recent five years of these journals are not available online on JSTOR. Also, JSTOR requires a license fee. If you access JSTOR from a Cornell IP address, then you are using the license Cornell has purchased. You cannot access JSTOR from a non-Cornell account.

Good course project: Read one or more of the studies of the EMH and prepare a report summarizing the work. The two review articles by Fama could help you find studies that would interest you. Using some new financial markets data, try to replicate some of original work.

3.13 Behaviorial Finance — a challenge to EMH

- behavioral finance combines psychology and economics
- it is relatively new, having started in the 1980's
- we will study behavioral finance later in Chapter 13
- the central conclusions of behaviorial finance are that
  - markets do have inefficiencies
  - one can sometimes make money by exploiting these inefficiencies

3.14 Continuous Compounding

One dollar invested for one year at a 5% rate of simple interest is worth \$1.05 at the end of one year. If instead the 5% interest is compounded semi-

N	$(D_1 - 1{,}050)$
1	0
2	.6250
4	.9433
12	1.1619
52	1.2458
365	1.2675
24*365	1.2709
60*24*365	1.2711
∞	1.2711

annually then the worth after one year is

$$\left(1 + \frac{.05}{2}\right)^2 = 1.050625.$$

If the compounding is daily, then the worth after one year is

$$\left(1 + \frac{.05}{365}\right)^{365} = 1.0512675.$$

If one compounded the interest every hour, then the worth after one year would be

$$\left(1 + \frac{.05}{(24)(365)}\right)^{(24)(365)} = 1.0512709.$$

As the compounding becomes more and more frequent, the worth after one year has a limit:

$$\lim_{N \to \infty} \left(1 + \frac{.05}{N}\right)^N = \exp(.05) = 1.0512711.$$

This limit is called "continuously compounding."

The following table shows the convergence to the limit. In the table N is the number of times that interest is compounded in a year and D_1 is the value at the end of the year of an initial deposit of one thousand dollars. The second column contains the values of $(D_1 - 1{,}050)$, which is the extra value (in dollars) of compounded over simple interest. To five significant digits, there is no difference between compounding every minute and continuous compounding.

In general, if D dollars are investing at a continuously compounded rate r for one year, then the value after one year is $\exp(r)\, D$. The return is

$\exp(r)$ so that the log return is r. This is the reason that the log return on an asset is called the continuously compounded rate.

Here is another way of looking at continuous compounding. Let t be time in years and let D_t to the value of a deposit that is growing according to the differential equation

$$\frac{d\,D_t}{d\,t} = r D_t. \tag{3.1}$$

Then the solution to this differential equation is

$$D_t = D_0 \exp(rt). \tag{3.2}$$

In particular, $D_1 = D_0 \exp(r)$.

3.15 Summary

Let P_t be the price of an asset at time t. Then P_t/P_{t-1} is the simple gross return and $R_t = P_t/P_{t-1} - 1$ is the simple net return. ("Simple" means one period.) The gross return over the last k periods is $1 + R_t(k) = P_t/P_{t-k}$. Let $p_t = \log(P_t)$. The (one-period) log return is $r_t = p_t - p_{t-1}$. $R_t \approx r_t$.

Log returns are often modeled as geometric random walks. The geometric random walk model implies that log returns are mutually independent; one cannot predict future returns from past returns. The model also implies that R_t is lognormally distributed.

Empirical research by Eugene Fama, Alfred Cowles, Holbrook Working, Maurice Kendall, and other econometricians supports the geometric random walk model.

The geometric random walk model suggested the efficient market hypothesis (EMH) that states that all valuable information is reflected in the market prices; price changes occur because of unanticipated new information. There are three forms of the EMH, the weak form, the semi-strong form, and the strong form.

3.16 References

Bernstein, P., (1996), *Against the Gods: The Remarkable Story of Risk,* John Wiley & Sons, New York.

Bernstein, P., (1992), *Capital Ideas: The Improbable Origins of Modern Wall Street*, Free Press, New York.

Chapter 4

Time Series Models: 4/11/02

4.1 Time Series

A univariate time series is a sequence of observations taken over time, for example, a sequence of daily returns on a stock. A multivariate time series is a sequence of *vectors* of observations taken in time, for example, the sequence of vectors of returns on a fixed set of stocks.

In this chapter, we will study statistical models for univariate times series. These models are widely used in econometrics as well as in other business and OR applications. For example, time series models are routinely used in OR to model the output of simulations and are used in business for forecasting.

4.2 Stationary Processes

Often we observe a time series whose fluctuations appear random but with the same type of random behavior from one time period to the next. For example, outside temperature has random and seasonal variation but each summer is similar to the past summers.[1] Also, interest rate in the past are similar to those of the present and we exhibit future interest rates also to be similar to those of the past. *Stationary stochastic processes* are probability models for such time series.

A process is stationary if its behavior is unchanged by shifts in time. A process is *weakly stationary* if its mean, variance, and covariance are unchanged by time shifts. More precisely, $X_1, X_2, \ldots$ is a weakly stationary

[1] Global warming might of course change this.

process if

- $E(X_i) = \mu$ (a constant) for all i
- Var(X_i) $= \sigma^2$ (a constant) for all i
- Corr(X_i, X_j) $= \rho(|i - j|)$ for all i and j for some function ρ

Thus, the mean and variance do not change with time and the correlation between two observations depends only on the time distance between them. For example, if the process is stationary then the correlation between X_2 and X_5 is the same as the correlation between X_7 and X_{10}, since each pair are separated from each other by three units of time.

ρ is called the correlation function of the process. Note that $\rho(h) = \rho(-h)$. Why?

The covariance between X_t and X_{t+h} is denoted by $\gamma(h)$. $\gamma(\cdot)$ is called the autocovariance function. Note that $\gamma(h) = \sigma^2\rho(h)$ and that $\gamma(0) = \sigma^2$ since $\rho(0) = 1$.

Many financial time series, for example, stock prices, do not appear stationary, but the *changes* in these time series do appear stationary and can be modeled as stationary processes. For this reason, stationary time series models are far more applicable that they might appear.

4.2.1 Weak White Noise

White noise is the simplest example of a stationary process. $X_1, X_2, \ldots$ is a *weak white noise process*, denoted WN(0,σ^2), if

- $E(X_i) = 0$ for all i
- Var(X_i) $= \sigma^2$ (a constant) for all i
- Corr(X_i, X_j) $= 0$ for all $i \neq j$

If in addition, $X_1, X_2 \ldots$ are independent normal random variables, then the process is called a *Gaussian white noise process*. (The normal distribution is sometimes called the Gaussian distribution.)

A weak white noise process is weakly stationary with

$$\begin{aligned} \rho(0) &= 1 \\ \rho(t) &= 0 \text{ if } t \neq 0 \end{aligned}$$

so that

$$\begin{aligned} \gamma(0) &= \sigma^2 \\ \gamma(t) &= 0 \text{ if } t \neq 0. \end{aligned}$$

Properties of Gaussian white noise

$$E(X_{i+t}|X_1,\ldots,X_i) = 0 \text{ for all } t \geq 1.$$

(You cannot predict the future, because the future is independent of the past and present.)

To us, "white noise" will mean weak white noise, which includes Gaussian white noise as a special case.

White noise (either weak or Gaussian) is uninteresting in itself but is the building block of important time series models used for economic data.

4.2.2 Estimating parameters of a stationary process

Suppose we observe $y_1,\ldots,y_n$ from a stationary process. To estimate the mean μ and variance σ^2 of the process we use the sample mean $\overline{y}$ and sample variance s^2 defined in equation (2.16).

To estimate the autocovariance function we use

$$\widehat{\gamma}(h) = n^{-1}\sum_{j=1}^{n-h}(y_{j+h}-\overline{y})(y_j-\overline{y}).$$

To estimate $\rho(\cdot)$ we use the sample autocorrelation function (SACF) defined as

$$\widehat{\rho}(h) = \frac{\widehat{\gamma}(h)}{\widehat{\gamma}(0)}.$$

4.3 AR(1) processes

Although white noise is not an appropriate model for a time series with correlation, time series models with correlation can be built out of white noise. The simplest stationary processes with correlation are autoregressive processes where y_t is modeled as a weighted average of past observations plus a white noise "error." In other words, autoregressive processes follow a regression model where y_t is the "response" or "outcome" and where past values of the process are the "independent" or "predictor" variables. We will start with AR(1) processes, the simplest autoregressive processes.

Let $\epsilon_1,\epsilon_2,\ldots$ be WH(0,σ_ϵ^2). We say that $y_1,y_2,\ldots$ is an AR(1) process if for some constant parameters μ and ϕ

$$y_t - \mu = \phi(y_{t-1}-\mu) \tag{4.1}$$

for all t.

If $|\phi| < 1$, then $y_1, \ldots$ is a weakly stationary process. Its mean is μ. Simple algebra shows that (4.1) can be rewritten as

$$y_t = (1 - \phi)\mu + \phi y_{t-1} + \epsilon_t. \tag{4.2}$$

Remember the linear regression model, $y_t = \beta_0 + \beta_1 x_t + \epsilon_t$ from your statistics courses or peak ahead to Chapter 6 for a review of regression analysis. Equation (4.2) is just a linear regression model with $\beta_0 = (1 - \phi)\mu$ and $\beta_1 = \phi$. If it is assumed that the process has a zero mean, that is, that $\mu = 0$, then $\beta_0 = 0$ as well. Linear regression with $\beta_0 = 0$ is the "linear regression through the origin model." The term *autoregression* refers to the regression of the process on its own past values.

When $|\phi| < 1$ then

$$y_t = \mu + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \cdots = \mu + \sum_{h=0}^{\infty} \phi^h \epsilon_{t-h} \tag{4.3}$$

Equation (4.3) is called the infinite moving average (MA(∞)) represention of the process. This equation shows that y_t is a weighted average of *all* past values of the white noise process. This representation should be compared to the AR(1) representation that shows y_t as depending on y_{t-1} and ϵ_t. Since $|\phi| < 1$, $\phi^h \to 0$ as the lag $h \to \infty$. Thus, the weights given to the distant past are small. In fact, they are quite small. For example, if $\phi = .5$ then $\phi^{10} = 0.00098$.

4.3.1 Properties of a stationary AR(1) process

When $|\phi| < 1$ (stationarity), then

1.
$$E(y_t) = \mu \quad \forall t$$

2.
$$\gamma(0) = \text{Var}(y_t) = \frac{\sigma_\epsilon^2}{1 - \phi^2} \quad \forall t$$

3.
$$\gamma(h) = \text{Cov}(y_t, y_{t+h}) = \frac{\sigma_\epsilon^2 \phi^{|h|}}{1 - \phi^2} \quad \forall t.$$

4.

$$\rho(h) = \text{Corr}(y_t, y_{t+h}) = \phi^{|h|} \quad \forall t.$$

It is important to remember that these formulas hold only if $|\phi| < 1$ and only for AR(1) processes. If $|\phi| \geq 1$, then the AR(1) process is nonstationary, and the mean, variance, and correlation are not constant.

These formulas can be proved using (4.3). For example using (2.15)

$$\text{Var}(y_t) = \text{Var}\left(\sum_{h=0}^{\infty} \phi^h \epsilon_{t-h}\right) = \sigma_\epsilon^2 \sum_{h=0}^{\infty} \phi^{2h} = \frac{\sigma_\epsilon^2}{1-\phi^2}.$$

Also, for $h > 0$

$$\text{Cov}\left(\sum_{i=0}^{\infty} \epsilon_{t-i}\phi^i, \sum_{j=0}^{\infty} \epsilon_{t+h-j}\phi^j\right) = \frac{\sigma_\epsilon^2 \phi^{|h|}}{1-\phi^2}.$$

Be sure to distinguish between σ_ϵ^2 which is the variance of the stationary white noise process $\epsilon_1, \epsilon_2, \ldots$ and $\gamma(0)$ which is the variance of the AR(1) process $y_1, y_2, \ldots$. We can see from the result above that $\gamma(0)$ is bigger than σ_ϵ^2 unless $\phi = 0$ in which case $y_t = \epsilon_t$.

4.3.2 Nonstationary AR(1) processes

Random Walk

If $\phi = 1$ then

$$y_t = y_{t-1} + \epsilon_t$$

and the process is **not** stationary. This is the random walk process we saw in Chapter 3.

It is easy to see that

$$y_t = y_0 + \epsilon_1 + \cdots \epsilon_t.$$

Suppose we start at the process at an arbitrary point y_0. Then $E(y_t|y_0) = y_0$ for all t, which is constant but depends entirely on the arbitrary starting point. Moreover, $\text{Var}(y_t|y_0) = t\sigma_\epsilon^2$ which is not stationary but rather increases linearly with time. The increasing variance makes the random walk "wander" in that y_t takes increasingly longer excursions away from its mean of y_0.

AR(1) processes when $|\phi| > 1$

When $|\phi| > 1$, an AR(1) process has explosive behavior. This can be seen in Figure 4.1. This figure shows simulations of 200 observations from AR(1) processes with various values of ϕ. The explosive case where $\phi = 1.02$ clearly is different than the other cases where $|\phi| \leq 1$. However, the case where $\phi = 1$ is not that much different than $\phi = .9$ even though the former is non-stationary while the latter is stationary.

The ability to distinguish the three types of AR(1) processes (stationary, random walk, and explosive) depends on the length of the observed series. For short AR(1), it is very difficult to tell if the process is stationary, random walk, or explosive. For example, in Figure 4.2, we see 30 observations from processes with the same parameter values as in Figure 4.1. Notice that unlike in Figure 4.1, in Figure 4.2 the random walk process with $\phi = 1$ appears very similar to the explosive process with $\phi = 1.02$.

If we observe the AR processes for longer than 200 observations, then the the behavior of $\phi = .9$ and $\phi = 1$ processes would not look as similar as in Figure 4.1. For example, in Figure 4.3 there are 1,000 observations from each of the processes. Now the processes with $\phi = .9$ and $\phi = 1$ look dissimilar, although they look similar in Figure 4.1 with only 200 observations. The stationary process with $\phi = .9$ continues to return to its mean of zero. The random walk ($\phi = 1$) wanders without tending to return to zero.

Suppose an explosive AR(1) process starts at $y_0 = 0$ and has $\mu = 0$. Then

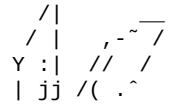
$$\begin{aligned} y_t &= \phi y_{t-1} + \epsilon_t = \phi(\phi y_{t-2} + \epsilon_{t-1}) + \epsilon_t = \phi^2 y_{t-2} + \phi\epsilon_{t-1} + \epsilon_t = \cdots \\ &= \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \cdots + \phi^{t-1}\epsilon_1 + \phi^t y_0. \end{aligned}$$

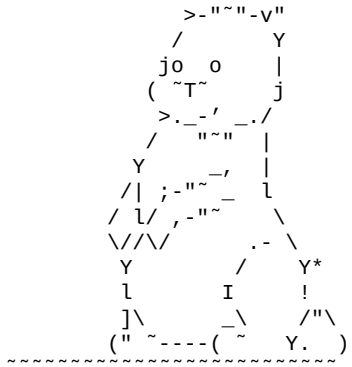
Therefore, $E(y_t) = \phi^t y_o$ and

$$\text{Var}(y_t) = \sigma^2(1 + \phi^2 + \phi^4 + \cdots + \phi^{2(t-1)}) = \sigma^2 \frac{\phi^{2t} - 1}{\phi - 1}.$$

Since $|\phi| > 1$, this variance increases geometrically fast at $t \to \infty$.

Explosive AR processes are not widely used in econometrics since economic growth is usually not explosive, though these processes may serve as good models of rabbit populations.





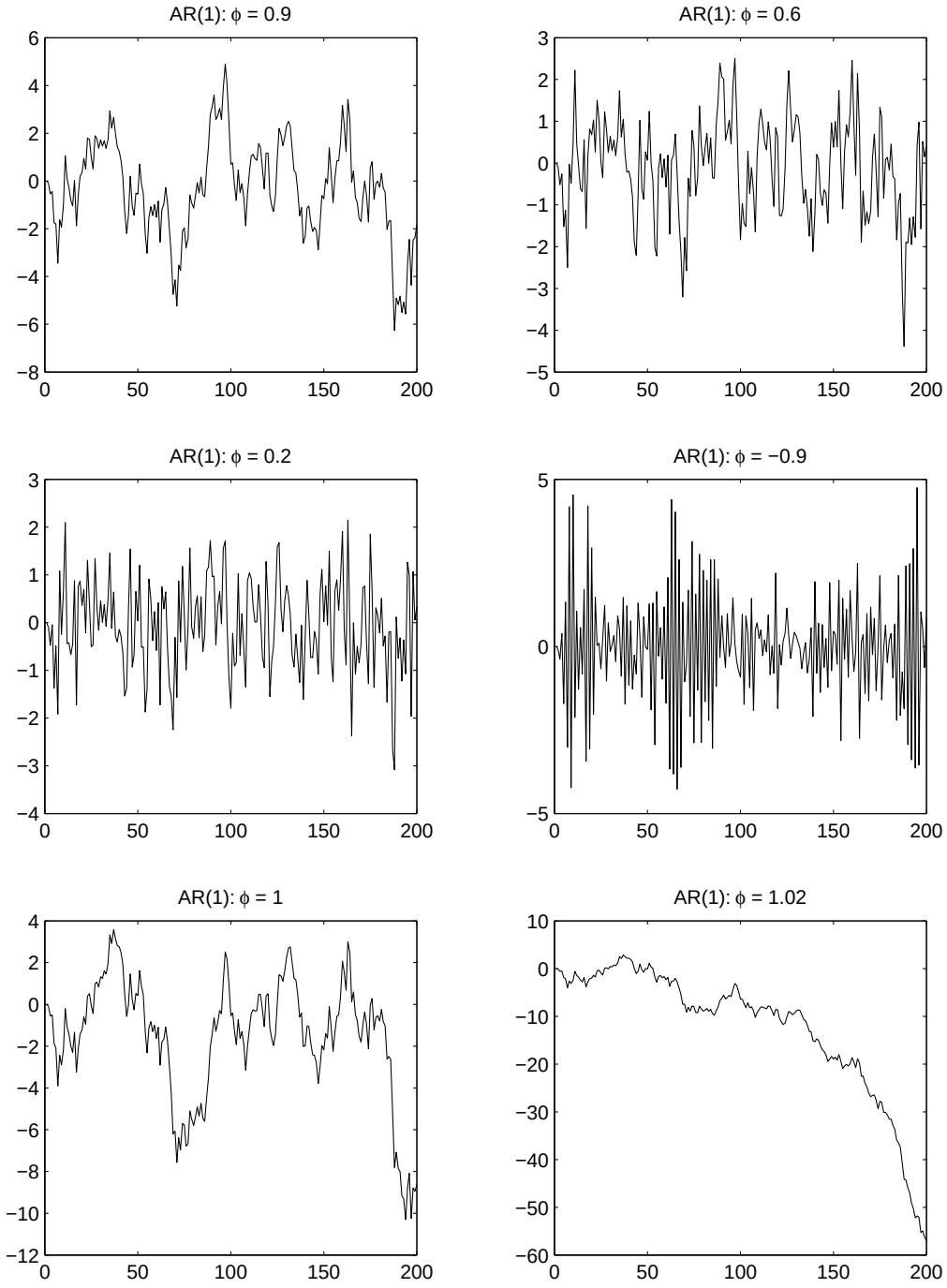


Figure 4.1: *Simulations of 200 observations from AR(1) processes with various values of ϕ and $\mu = 0$. The white noise "residual" or "error" process $\epsilon_1, \epsilon_2, \ldots$ is the same for all six AR(1) processes.*

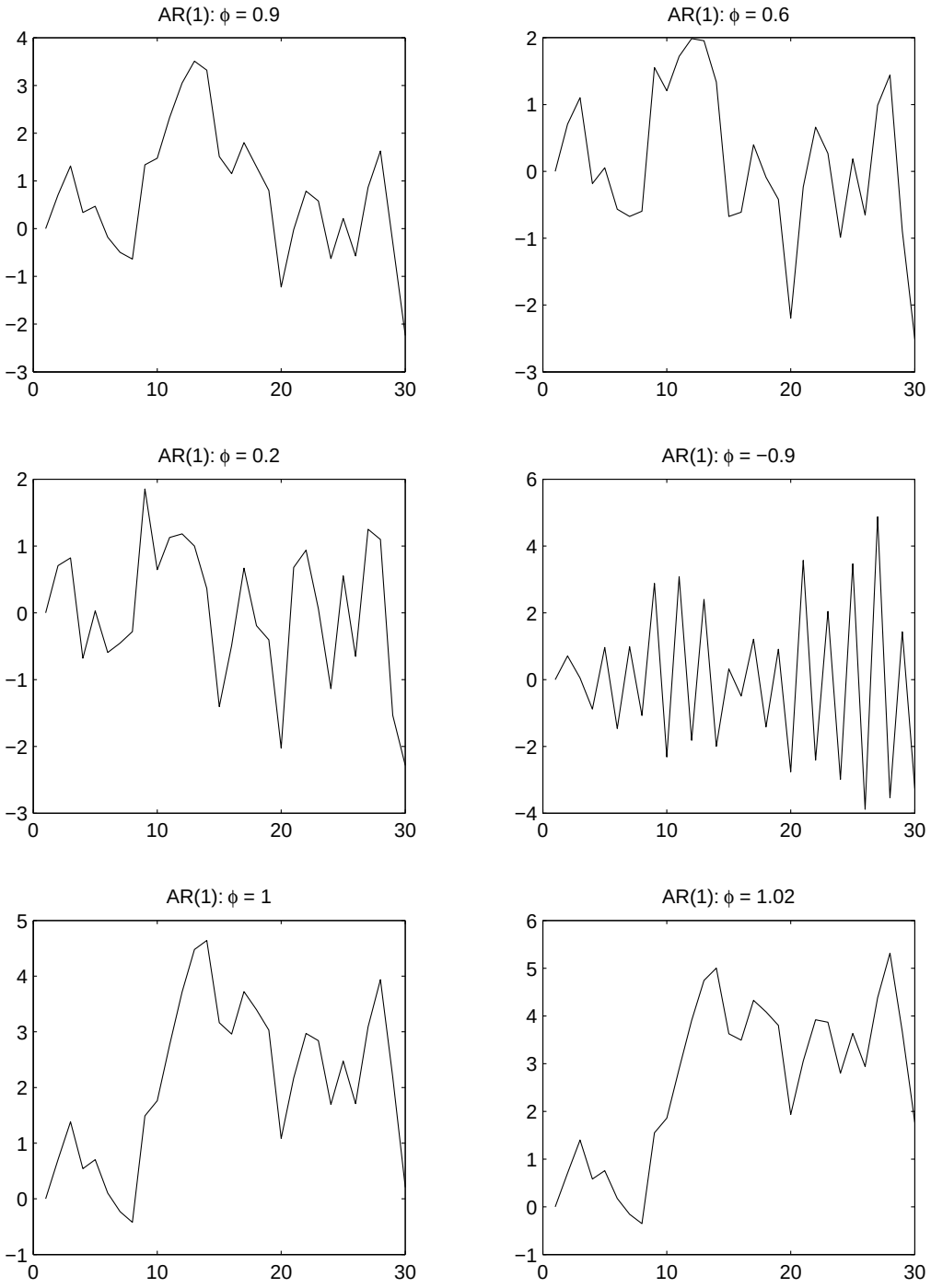


Figure 4.2: *Simulations of 30 observation from AR(1) processes with various values of ϕ and $\mu = 0$. The white noise "residual" or "error" process $\epsilon_1, \epsilon_2, \ldots$ is the same for all six AR(1) processes.*

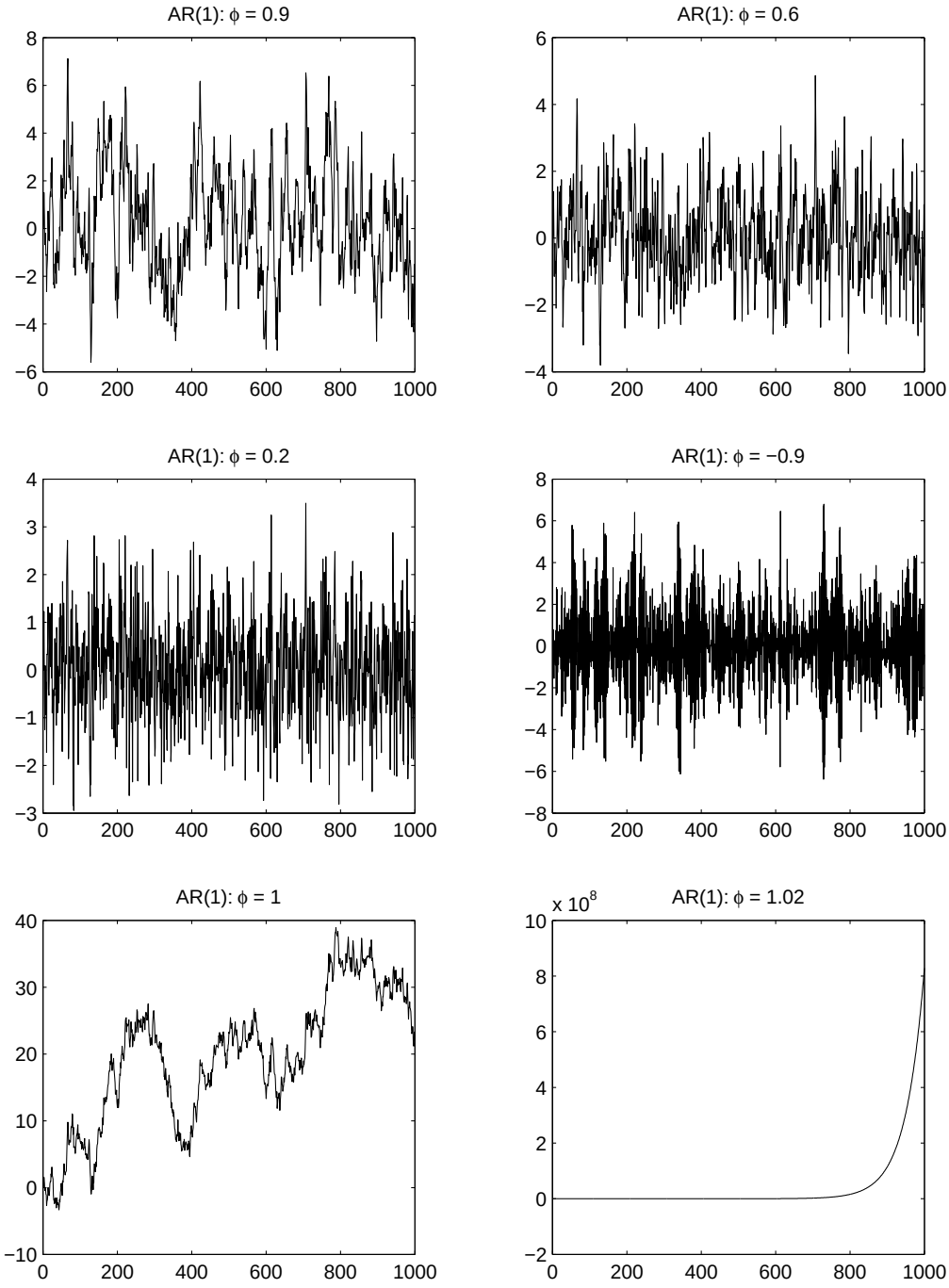


Figure 4.3: *Simulations of 1000 observation from AR(1) processes with various values of ϕ and $\mu = 0$. The white noise "residual" or "error" process $\epsilon_1, \epsilon_2, \ldots$ is the same for all six AR(1) processes.*

4.3.3 Estimation

Depending upon the application, one will want to fit an AR(1) to either one of the variables in the raw data or a variable that has been constructed from the raw data. In finance applications, one often has the prices as the raw data but wants to fit an AR(1) to the log returns. To create the log returns, one first log-transforms the prices and then differences the log prices. Both MINITAB and SAS have functions to take logarithms and to do differencing. For example, to take differences in MINITAB, go to the "Stat" menu, then the "Time Series" menu, and then select "differences." Once a variable containing the log returns has been created, one then can fit an AR(1) model to it.

Let's assume we have a time series $y_1, \ldots, y_n$ and we want to fit an AR(1) model to this series. Since an AR(1) model is a linear regression model, it can be analyzed using linear regression software. One creates a lagged variable in y_t and uses this as the "x-variable" in the regression. MINITAB and SAS both support lagging. For example, in MINITAB, go to the "Stat" menu, then the "Time Series" menu, and then select "lag."

The least squares estimation of ϕ and μ minimize

$$\sum_{t=2}^{n} \Big[\{y_t - \mu\} - \{\phi(y_{t-1} - \mu)\}\Big]^2.$$

If the errors $\{\epsilon_1, \ldots, \epsilon_n\}$ are *Gaussian* white noise then the least-squares estimate is also the MLE.

Moreover, both MINITAB or SAS have special procedures for fitting AR models.

In MINITAB, go the the "Stat" menu, then the "Time Series" menu, and then choose ARIMA. Use 1 autoregressive parameter, 0 differencing, and 0 moving average parameters.

In SAS, use the "AUTOREG" or the "ARIMA" procedure.

4.3.4 Residuals and checking the model

Once μ and ϕ have been estimated, one can estimate the white noise process $\epsilon_1, \ldots, \epsilon_n$. Rearranging equation (4.1) we have

$$\epsilon_t = (y_t - \mu) - \phi(y_{t-1} - \mu).$$

The residuals, $\widehat{\epsilon}_1, \widehat{\epsilon}_2, \ldots, \widehat{\epsilon}_n$, defined by

$$\widehat{\epsilon}_t = (y_t - \widehat{\mu}) - \widehat{\phi}(y_{t-1} - \widehat{\mu})$$

estimate $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and can be used to check the assumption that $y_1, y_2, \ldots, y_n$ is an AR(1) process; any autocorrelation in the residuals is evidence against the assumption of an AR(1) process.

To appreciate why residual autocorrelation indicates a possible problem with the model, suppose that we are fitting an AR(1) model but the true model is a AR(2) process[2] given by

$$(y_t - \mu) = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \epsilon_t.$$

Since we are fitting the wrong model, there is no hope of estimating ϕ_2. Moreover, $\widehat{\phi}$ does not necessarily estimate ϕ_1 because of bias caused by model misspecification. Let ϕ^* be the expected value of $\widehat{\phi}$. For the purpose of illustration, assume that $\widehat{\mu} \approx \mu$ and $\widehat{\phi} \approx \phi^*$. This is a sensible approximation if the sample size n is large enough. Then

$$\begin{aligned} \widehat{\epsilon}_t &\approx (y_t - \mu) - \phi^*(y_{t-1} - \mu) \\ &= \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \epsilon_t - \phi^*(y_{t-1} - \mu) \\ &= (\phi_1 - \phi^*)(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \epsilon_t. \end{aligned}$$

Thus, the residuals do not estimate the white noise process as they would if the correct AR(2) model were used. Even if there is no bias in the estimation of ϕ so that $\phi_1 = \phi^*$ and so that the term $(\phi_1 - \phi^*)(y_{t-1} - \mu)$ drops out, the presence of $\phi_2(y_{t-2} - \mu)$ in the residuals causes the residuals to be autocorrelated.

To test for residual autocorrelation one can use the "test bounds" provided by MINITAB's or SAS's autocorrelation plots. One can also use the Ljung-Box test of the null hypothesis that simultaneously tests that all autocorrelations up to a specified lag are zero.

Example: GE daily returns

Autoregressive models can be analyzed in both MINITAB and SAS.

The MINITAB output was obtained by running MINITAB interactively.

Here is the MINITAB output. The variable "logR" is the time series of log returns.

[2]We will discuss higher order AR models soon.

```
   2/2/01 10:45:25 AM

Welcome to Minitab, press F1 for help.
Retrieving worksheet from file: C:\COURSES\OR473\MINITAB\GE_DAILY.MTW
# Worksheet was saved on Wed Jan 10 2001

Results for: GE_DAILY.MTW

ARIMA Model: logR

ARIMA model for logR

Estimates at each iteration
Iteration        SSE     Parameters
    0         2.11832    0.100    0.090
    1         0.12912    0.228    0.015
    2         0.07377    0.233    0.001
    3         0.07360    0.230    0.000
    4         0.07360    0.230   -0.000
    5         0.07360    0.230   -0.000
Relative change in each estimate less than  0.0010

Final Estimates of Parameters
Type          Coef     SE Coef         T        P
AR   1      0.2299      0.0621      3.70    0.000
Constant -0.000031    0.001081     -0.03    0.977
Mean     -0.000040    0.001403

Number of observations:  252
Residuals:    SS =  0.0735911  (backforecasts excluded)
              MS = 0.0002944  DF = 250

Modified Box-Pierce (Ljung-Box) Chi-Square statistic
Lag               12        24        36        48
Chi-Square      23.0      33.6      47.1      78.6
DF                10        22        34        46
P-Value        0.011     0.054     0.066     0.002
```

An AR(p) model can be fit SAS using either the AUTOREG procedure or the ARIMA procedure. One difference between the two procedures is that AUTOREG outputs the estimated "constant" in the model (what we called β_0 earlier) whereas ARIMA outputs the estimated mean.

The SAS output below comes from running the following program that fits an AR(1) model using AUTOREG.

```
options linesize = 72 ;
comment Restrict the linesize to 72 characters ;
data ge ;  comment Start the data step ;
infile 'c:\courses\or473\data\ge.dat' ;
comment Specify the input data set ;
input close ;
comment Create a new variable ;
D_p = dif(close);
comment Take first differences ;
logP = log(close) ;
logR = dif(logP) ;
comment logR = log returns ;
run ;
title 'GE - Daily prices, Dec 17, 1999 to Dec 15, 2000' ;
title2 'AR(1)' ;
proc autoreg ;
model logR =/nlag = 1 ;
run ;
```

Here is the SAS output.

```
          GE - Daily prices, Dec 17, 1999 to Dec 15, 2000           1
                               AR(1)    10:32 Friday, February 2, 2001

                        The AUTOREG Procedure

                      Dependent Variable    logR


                   Ordinary Least Squares Estimates

     SSE                 0.07762133    DFE                     251
     MSE                  0.0003092    Root MSE            0.01759
     SBC                 -1316.8318    AIC                -1320.3612
     Regress R-Square        0.0000    Total R-Square         0.0000
     Durbin-Watson           1.5299


                                    Standard                  Approx
  Variable        DF     Estimate        Error    t Value    Pr > |t|

  Intercept        1    -0.000011     0.001108      -0.01      0.9917


                    Estimates of Autocorrelations

                   Lag    Covariance     Correlation

                     0      0.000308        1.000000
                     1      0.000069        0.225457

                    Estimates of Autocorrelations

          Lag    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

            0    |                   |********************|
            1    |                   |*****               |


                    Preliminary MSE    0.000292


                Estimates of Autoregressive Parameters

                                    Standard
              Lag     Coefficient          Error    t Value

                 1       -0.225457        0.061617      -3.66
          GE - Daily prices, Dec 17, 1999 to Dec 15, 2000           2
                               AR(1)    10:32 Friday, February 2, 2001

                        The AUTOREG Procedure

                        Yule-Walker Estimates
```

SSE	0.07359998	DFE	250
MSE	0.0002944	Root MSE	0.01716
SBC	-1324.6559	AIC	-1331.7148
Regress R-Square	0.0000	Total R-Square	0.0518
Durbin-Watson	1.9326		

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
Intercept	1	-0.000040	0.001394	-0.03	0.9773

From MINITAB we see that $\widehat{\phi} = .2299$ and the estimated standard deviation of $\widehat{\phi}$ is 0.0621. The t-value for testing $\mathrm{H}_0 : \ \phi = 0$ versus $\mathrm{H}_1 : \ \phi \neq 0$ is $.2299/.0621 = 3.70$ and the p-value is .000 (zero to three decimals). Since the p-value is so small, we reject the null hypothesis.

[Note: Recall from Section 2.6.1 that small p-values are significant; we reject the null hypothesis if the p-value is less than α, e.g., less than .05.]

The null hypothesis is that the log returns are white noise and the alternative is that they are correlated. Thus, we have evidence against the geometric random walk hypothesis. However, $\phi = .2299$ is not large. Since $\rho(h) = \phi^h$, the correlation between successive log returns is .2299 and the squared correlation is only .0528 — as discussed in Section 2.2.1 the squared correlation is the fraction of variation that is predictable so we see that only about five percent of the variation in a log return can be predicted by the previous day's return.

We have seen that an AR(1) process fits the GE log returns better than a white noise model. Of course, this is not proof that the AR(1) fits these data, only that it fits better than a white noise model. To check that the AR(1) fits well, one looks at the sample autocorrelation function (SACF) of the residuals. A plot of the residual SACF can be requested when using either MINITAB or SAS.

The SACF of the residuals from the GE daily log returns shows high negative autocorrelation at lag 6; $\widehat{\rho}(6)$ is outside the test limits so is "significant" at $\alpha = .05$; see Figure 4.4. This is disturbing.

Moreover, the more conservate Ljung-Box "simultaneous" test that $\rho(1) = \cdots \rho(12) = 0$ has $p = .011$. Since the AR(1) model does not fit well, one might consider more complex models. These will be discussed in the following sections.

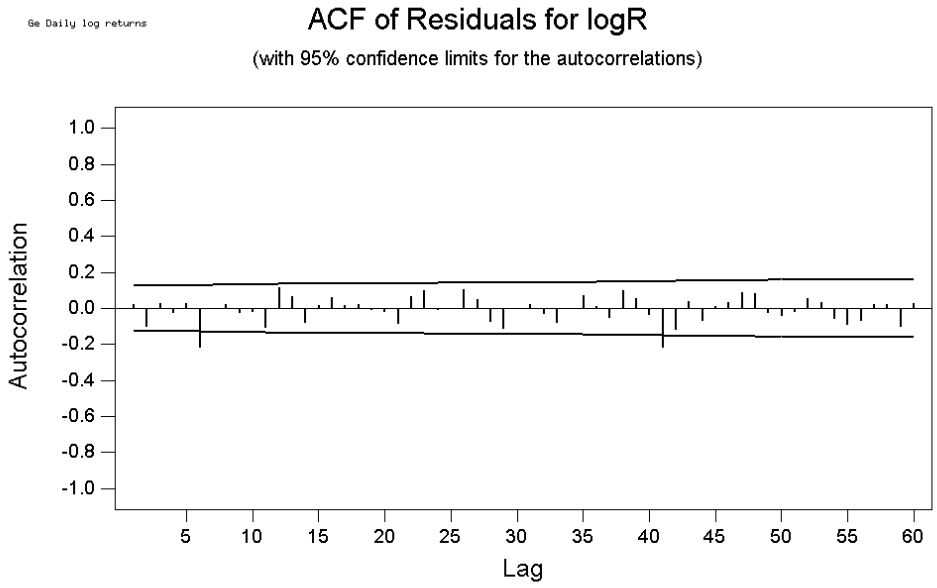


Figure 4.4: *SACF of residuals from an AR(1) fit to the GE daily log returns. Notice the large negative residual autocorrelation at lag 6. This is a sign that the AR(1) model does not fit well.*

The SAS estimate of ϕ is $-.2254$. SAS uses the model

$$y_t = \beta_0 - \phi y_{t-1} + \epsilon_t \tag{4.4}$$

so SAS's ϕ is the negative of ϕ as we, and MINITAB, define it. The difference, .2299 versus .2254, between MINITAB and SAS is due to slight variation in the estimation algorithm.

We can also estimate μ and test that μ is zero. From the MINITAB output, we see $\widehat{\mu}$ is nearly zero, the t-value for testing that μ is zero is very small while the p-value is near one. Remember that *small* values of the p-value are significant; since the p-value is *large* we accept the null hypothesis that μ is zero.

The parameter β_0 in equation (4.4) is called the "constant" or "intercept" of the model and equals $(1 - \phi)\mu$. Since $\phi < 1$ for a stationary process, $\mu = 0$ if and only if β_0 is zero. Therefore, the p-value in MINITAB output for testing that the "constant" is zero is also the p-value for testing that the mean is zero.

4.4 AR(p) models

y_t is an AR(p) process if

$$(y_t - \mu) = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \epsilon_t$$

where $\epsilon_1, \ldots, \epsilon_n$ is WN$(0, \sigma_\epsilon^2)$.

This is a multiple linear regression model with lagged values of the time series as the "x-variables." The model can be reexpressed as

$$y_t = \beta_0 + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + \epsilon_t,$$

where $\beta_0 = \{1 - (\phi_1 + \ldots + \phi_p)\}\mu$. The parameter β_0 is called the "constant" or "intercept" as in an AR(1) model. $\{1 - (\phi_1 + \ldots + \phi_p)\} > 0$ for a stationary process, so $\mu = 0$ if and only if β_0 is zero. Therefore, analogously as for an AR(1) model, the p-value in MINITAB output for testing that the "constant" is zero is also the p-value for testing that the mean is zero.

The least-squares estimator minimizes

$$\sum_{t=p+1}^{n} \{y_t - (\beta_0 + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p})\}^2.$$

The least-squares estimator can be calculated using a multiple linear regression program but one must create "x-variables" by lagging the time series with lags 1 throught p. It is easier to use the ARIMA command in MINITAB or SAS or SAS's AUTOREG procedure; these procedures do the lagging automatically.

4.4.1 Example: GE daily returns

The SAS program shown above was rerun with

```
model logR =/nlag = 1
```

replaced by

```
model logR =/nlag = 6 .
```

The output is on the course's web site as "GE DAILY, AR(6) (SAS)."

The autoregression coefficients (the ϕ_i) are "significant" at lags 1 and 6 but not at lags 2 through 5. Here "significant" means at $\alpha = .05$ which corresponds to absolute t-value bigger than 2. MINITAB will not allow $p > 5$ but SAS does not have such a constraint.

4.5 Moving Average (MA) Processes

4.5.1 MA(1) processes

The moving average process of order [MA(1)] is

$$y_t - \mu = \epsilon_t - \theta\epsilon_{t-1},$$

where as before the ϵ_t's are WH$(0, \sigma_\epsilon^2)$.

One can show that

$$E(y_t) = \mu,$$

$$\text{Var}(y_t) = \sigma_\epsilon^2(1 + \theta^2),$$

$$\gamma(1) = -\theta\sigma_\epsilon^2,$$

$$\gamma(h) = 0 \text{ if } |h| > 1,$$

$$\rho(1) = \frac{-\theta}{1 + \theta^2},$$

and

$$\rho(h) = 0 \text{ if } |h| > 1.$$

4.5.2 General MA processes

The MA(q) process is

$$y_t = \mu + \epsilon_t - \theta_1\epsilon_{t-1} - \cdots - \theta_q\epsilon_{t-q}. \tag{4.5}$$

One can show that $\gamma(h) = 0$ and $\rho(h) = 0$ if $|h| > q$. Formulas for $\gamma(h)$ and $\rho(h)$ when $|h| \leq q$ are given in time series textbooks but can be complicated and will not be needed by us.

Unlike AR(p) models where the "constant" in the model is not the same as the mean, in an MA(q) model μ, the mean of the process, is the same as β_0, the "constant" in the model. This fact can be appreciated by examining the right hand side of equation (4.5) where μ is the "intercept" or "constant" in the model and is also the mean of y_t because $\epsilon_t, \ldots, \epsilon_{t-q}$ all have mean zero.

4.6 ARIMA Processes

Stationary time series with complex autocorrelation behavior are better modeled by mixed autoregressive and moving average (ARMA) processes than by either a pure AR or pure MA process. ARIMA (autoregressive, integrated, moving average) processes are based on ARMA processes and are models for nonstationary time series.

ARIMA processes are more easily described if we introduce the "backwards" operator, B.

4.6.1 The backwards operator

The backwards operator B is defined by

$$B\, y_t = y_{t-1}$$

and, more generally,

$$B^k\, y_t = y_{t-k}.$$

Note that $B\, c = c$ for any constant c since a constant does not change with time.

4.6.2 ARMA Processes

The ARMA(p, q) process satisfies the equation

$$(1 - \phi\, B - \cdots - \phi_p\, B^p)(y_t - \mu) = (1 - \theta_1\, B - \ldots - \theta_q\, B^q)\epsilon_t. \tag{4.6}$$

A white noise process is ARMA(0,0) with $\mu = 0$ since if $p = q = 0$, then (4.6) reduces to

$$(y_t - \mu) = \epsilon_t.$$

4.6.3 The differencing operator

The differencing operator is $\Delta = 1 - B$ so that

$$\Delta y_t = y_t - B\, y_t = y_t - y_{t-1}.$$

Thus, differencing a time series produces a new time series consisting of the changes in the original series. For example, if $p_t = \log(P_t)$ is the log price, then the log return is

$$r_t = \Delta p_t.$$

Differencing can be iterated. For example,

$$\begin{aligned}\Delta^2 y_t = \Delta(\Delta y_t) &= \Delta(y_t - y_{t-1}) = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\ &= y_t - 2y_{t-1} + y_{t-2}.\end{aligned}$$

4.6.4 From ARMA processes to ARIMA process

Often the first or second differences of nonstationary time series are stationary. For example, the first differences of random walk (nonstationary) are white noise (stationary).

A time series y_t is said to be ARIMA(p, d, q) if $\Delta^d y_t$ is ARMA(p, q). For example, if log returns (r_t) on an asset are ARMA(p, q), then the log prices (p_t) are ARIMA($p, 1, q$).

The ARIMA procedures in MINITAB and SAS allow one to specify p, d, and q.

Notice that an ARIMA($p, 0, q$) model is the same as an ARMA(p, q) model. ARIMA(($p, 0, 0$), ARMA($p, 0$), and AR(p) models are the same. Also, ARIMA$(0, 0, q)$, ARMA$(0, q)$, and MA(q) models are the same. A random walk is an ARIMA$(0, 1, 0)$ model. Why?

The inverse of differencing is "integrating." The integral of a process y_t is the process w_t where

$$w_t = w_{t_0} + y_{t_0} + y_{t_0+1} + \cdots y_t.$$

where t_0 is an arbitrary starting time point and w_{t_0} is the starting value of the w_t process.

Figure 4.5 shows an AR(1), its "integral" and its "second integral," meaning the integral of its integral.

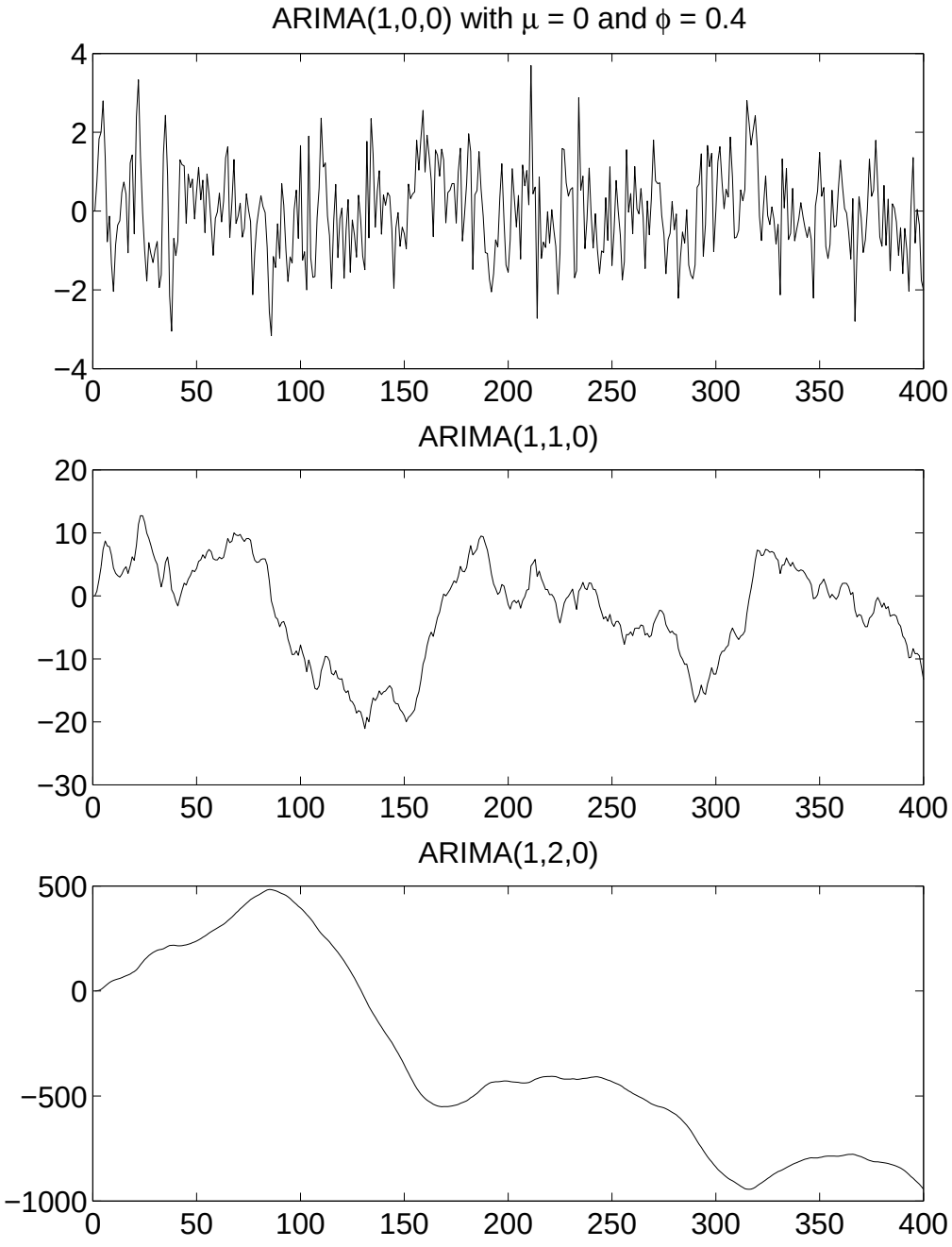


Figure 4.5: *The top plot is of an AR(1) process with* $\mu = 0$ *and* $\phi = 0.4$*. The middle and bottom plots are, respectively, the integral and second integral of this AR(1) process.*

4.7 Model Selection

Once the parameters $p, d,$ and q of an ARIMA process have been selected, the AR and MA coefficients can be estimated by maximum likelihood. But how do we choose $p, d,$ and q?

Generally, d is either 0, 1, or 2 and is chosen by looking at the SACF of y_t, Δy_t, and $\Delta^2 y_t$.

A sign that a process is nonstationary is that its SACF decays to zero very slowly. If this is true of y_t then the original series is nonstationary and should be differenced at least once.

If the SACF of Δy_t looks stationary then we use $d = 1$. Otherwise, we look at the SACF of $\Delta^2 y_t$; if this looks stationary we use $d = 2$.

I have never seen a real time series where $\Delta^2 y_t$ did not look stationary, but if one were encountered then $d > 2$ would be used.

Once d has been chosen, we know that we will fit an ARMA(p, q) process to $\Delta^d y_t$, but we still need to select p and q. This can be done by comparing various choices of p and q by some criterion that measures how well a model fits the data.

4.7.1 AIC and SBC

AIC and SBC are model selection criteria based on the log-likelihood.

Akaike's information criterion (AIC) is defined as

$$-2\log(L) + 2(p+q),$$

where L is the likelihood evaluated at the MLE.

Schwarz's Bayesian Criterion (SBC) is also called the Bayesian Information Criterion (BIC) and is defined as

$$-2\log(L) + \log(n)(p+q),$$

where n is the length of the time series.

The "best" model according to either criterion is the model that minimizes that criterion.

Either criteria will tend to select models with large values of the likelihood; this makes perfect sense since a large value of L means that the observed data are likely under that model.

The term $2(p+q)$ in AIC or $\log(n)(p+q)$ is a penalty on having too many parameters. Therefore, AIC and SBC both try to tradeoff a good fit to the data measured by L with the desire to use as few parameters as possible.

Note that $\log(n) > 2$ if $n \geq 8$. Since most time series are much longer than 8, SBC penalizes $p + q$ more than AIC. Therefore, AIC will tend to choose models with more parameters than SBC. Compared to SBC, with AIC the tradeoff is more in favor of a large value of L than a small value of $p + q$.

This difference between AIC and SBC is due to the way they were designed. AIC is designed to select the model that will predict best and is less concerned with having a few too many parameters. SBC is designed to select the true values of p and q exactly.

In practice the best AIC model is usually close to the best SBC model and often they are the same model.

Two model can be compared by likelihood ratio testing when one model is "bigger" than the other. Therefore, AIC and SBC are closely connected with likelihood ratio tests.

Unfortunately, MINITAB does not compute AIC and SBC and you can only use SAS if you have access to one of the ORIE computer labs.

Here's how you can calculate approximate AIC and SBC values using MINITAB. It can be shown that $\log(L) \approx (-n/2)\log(\widehat{\sigma}^2) + K$ where K is a constant that does not depend on the model of on the parameters. Since we only want to minimize AIC and SBC, the exact value of K is irrelevant and we will drop K. Thus, you can use the approximations

$$\text{AIC} \approx n\log(\widehat{\sigma}^2) + 2(p+q),$$

and

$$\text{SBC} \approx n\log(\widehat{\sigma}^2) + \log(n)(p+q).$$

$\widehat{\sigma}^2$ is called MSE (mean squared error) on the MINITAB output.

4.7.2 Stepwise regression applied to AR processes

Stepwise regression is a way of looking at a variety of regression models to see which ones fit the data well. Stepwise regression will be discussed briefly in Section 6.4. In backwards regression, sometimes called backstepping, one starts with all possible x-variables and eliminates them one at time until all remaining variables are "significant" by some criterion.

Stepwise regression can, of course, be applied to AR models since these are a type of multiple regression model. SAS's AUTOREG procedure allows backstepping as an option.

The following SAS program starts with an AR(6) model and backsteps.

```
options linesize = 72 ;
comment Restrict the linesize to 72 characters ;
data ge ;  comment Start the data step ;
infile 'c:\courses\or473\data\ge_quart.dat' ;
comment Specify the input data set ;
input close ;
D_p = dif(close);
comment Take first differences ;
logP = log(close) ;
logR = dif(logP) ;
comment logR = log returns ;
run ;
title 'GE - Quarterly closing prices, Dec 1900 to Dec 2000' ;
title2 'AR(6) with backstepping' ;
proc autoreg ;
model logR =/nlag = 6  backstep ;
run ;
```

Here is the SAS output:

```
      GE - Quarterly closing prices, Dec 1900 to Dec 2000          1
                   AR(6) with backstepping
                                  23:32 Tuesday, January 30, 2001

                    The AUTOREG Procedure

                   Dependent Variable    logR


                 Ordinary Least Squares Estimates

   SSE                 0.15125546    DFE                       38
   MSE                    0.00398    Root MSE              0.06309
   SBC                -102.20076    AIC                -103.86432
   Regress R-Square       0.0000    Total R-Square         0.0000
   Durbin-Watson          2.0710


                                  Standard                  Approx
Variable        DF     Estimate        Error    t Value    Pr > |t|

Intercept        1       0.0627       0.0101       6.21      <.0001


                  Estimates of Autocorrelations

                 Lag    Covariance     Correlation

                   0       0.00388        1.000000
                   1      -0.00014       -0.036627
                   2      -0.00023       -0.059114
                   3       0.00152        0.392878
                   4      -0.00014       -0.035792
                   5      -0.00075       -0.193269
                   6      0.000337        0.086919


                  Estimates of Autocorrelations

          Lag    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

            0    |                    |********************|
            1    |                   *|                    |
            2    |                   *|                    |
            3    |                    |********            |
            4    |                   *|                    |
            5    |                 ****|                    |
            6    |                    |**                  |


      GE - Quarterly closing prices, Dec 1900 to Dec 2000          2
                   AR(6) with backstepping
                                  23:32 Tuesday, January 30, 2001

                    The AUTOREG Procedure
```

Backward Elimination of
Autoregressive Terms

Lag	Estimate	t Value	Pr > |t|
4	0.020648	0.12	0.9058
2	-0.023292	-0.14	0.8921
1	0.035577	0.23	0.8226
6	0.082465	0.50	0.6215
5	0.170641	1.13	0.2655

Preliminary MSE 0.00328

Estimates of Autoregressive Parameters

Lag	Coefficient	Standard Error	t Value
3	-0.392878	0.151180	-2.60

Expected
Autocorrelations

Lag	Autocorr
0	1.0000
1	0.0000
2	0.0000
3	0.3929

Yule-Walker Estimates

SSE	0.12476731	DFE	37
MSE	0.00337	Root MSE	0.05807
SBC	-105.5425	AIC	-108.86962
Regress R-Square	0.0000	Total R-Square	0.1751
Durbin-Watson	1.9820		

GE - Quarterly closing prices, Dec 1900 to Dec 2000 3
AR(6) with backstepping
23:32 Tuesday, January 30, 2001

The AUTOREG Procedure

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
Intercept	1	0.0632	0.0146	4.33	0.0001

Expected

```
Autocorrelations

Lag     Autocorr

  0       1.0000
  1       0.0000
  2       0.0000
  3       0.3929
```

4.7.3 Using ARIMA in SAS: Cree data

Daily returns of Cree from December 1999 to December 2000 are shown in Figure 4.6.

In this example, we will illustrate fitting an ARMA model in SAS. We use daily log returns on Cree from December 1999 to December 2000. The SAS program is:

```
options linesize = 72 ;
data cree ;
infile 'U:\courses\473\data\cree_daily.dat ' ;
input month day year volume high low close ;
logP = log(close) ;
logR = dif(logP) ;
run ;
title Cree daily log returns ;
title2 ARMA(1,1) ;
proc arima ;
identify var=logR ;
estimate p=1 q=1 ;
run ;
```

The "identify" statement specifies the input series and tells SAS to compute the SACF. It can also be used to specify the amount of differencing; "identify var=logP(1) ;" would tell SAS to use the first differences of the log prices as input.

Here is the SAS output. The result is that the Cree log returns appear to be white noise since ϕ_1 (denoted by AR1,1 in SAS), θ_1 (denoted by MA1,1) and μ not significantly different from zero.

```
Cree daily log returns                          1
     ARMA(1,1)  15:18 Friday, February 2, 2001

The ARIMA Procedure
```

Name of Variable = logR

Mean of Working Series	-0.00071
Standard Deviation	0.067473
Number of Observations	252

Autocorrelations

Lag	Covariance	Correlation	-1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
0	0.0045526	1.00000	|********************|
1	0.00031398	0.06897	| . |* . | . | . | . | . | . | . | . | . | . |
2	-0.0000160	-.00351	| . | . | . | . | . | . | . | . | . | . | . |
3	-5.5958E-6	-.00123	| . | . | . | . | . | . | . | . | . | . | . |
4	-0.0002213	-.04862	| . | *. | . | . | . | . | . | . | . | . | . |
5	0.00002748	0.00604	| . | . | . | . | . | . | . | . | . | . | . |
6	-0.0000779	-.01712	| . | . | . | . | . | . | . | . | . | . | . |
7	-0.0000207	-.00454	| . | . | . | . | . | . | . | . | . | . | . |
8	-0.0003281	-.07207	| . | *. | . | . | . | . | . | . | . | . | . |
9	0.00015664	0.03441	| . | . |* . | . | . | . | . | . | . | . | . |
10	0.00057077	0.12537	| . | . |*** | . | . | . | . | . | . | . | . |
11	0.00023632	0.05191	| . | . |* . | . | . | . | . | . | . | . | . |
12	-0.0003475	-.07633	| . |**. | . | . | . | . | . | . | . | . | . |
13	-0.0001348	-.02961	| . | *. | . | . | . | . | . | . | . | . | . |
14	-0.0005590	-.12278	| . |**. | . | . | . | . | . | . | . | . | . |
15	0.00023425	0.05145	| . | . |* . | . | . | . | . | . | . | . | . |
16	-0.0001021	-.02242	| . | . | . | . | . | . | . | . | . | . | . |
17	-0.0000582	-.01278	| . | . | . | . | . | . | . | . | . | . | . |
18	-0.0007147	-.15699	| . |***| . | . | . | . | . | . | . | . | . |
19	0.00006314	0.01387	| . | . | . | . | . | . | . | . | . | . | . |
20	-0.0000466	-.01024	| . | . | . | . | . | . | . | . | . | . | . |
21	-0.0001681	-.03692	| . | *. | . | . | . | . | . | . | . | . | . |
22	-0.0001439	-.03161	| . | *. | . | . | . | . | . | . | . | . | . |
23	-0.0002135	-.04690	| . | *. | . | . | . | . | . | . | . | . | . |
24	0.00007502	0.01648	| . | . | . | . | . | . | . | . | . | . | . |

"." marks two standard errors

Cree daily log returns 2
ARMA(1,1) 15:18 Friday, February 2, 2001

The ARIMA Procedure

Inverse Autocorrelations

Lag	Correlation	-1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
1	-0.11452	| . |**. | . | . | . | . | . | . | . | . | . |
2	0.06356	| . | . |* . | . | . | . | . | . | . | . | . |
3	-0.08905	| . |**. | . | . | . | . | . | . | . | . | . |
4	0.12788	| . | . |***| . | . | . | . | . | . | . | . |
5	-0.04576	| . | *. | . | . | . | . | . | . | . | . | . |
6	0.07209	| . | . |* . | . | . | . | . | . | . | . | . |

```
  7       -0.06322    |                 . *|  .                |
  8        0.09828    |                 .  |**.               |
  9       -0.04639    |                 . *|  .                |
 10       -0.05006    |                 . *|  .                |
 11       -0.09283    |                 .**|  .                |
 12        0.10049    |                 .  |**.               |
 13       -0.02141    |                 .  |  .                |
 14        0.15284    |                 .  |***               |
 15       -0.09318    |                 .**|  .                |
 16        0.05864    |                 .  |* .               |
 17       -0.02983    |                 . *|  .                |
 18        0.16300    |                 .  |***               |
 19       -0.05602    |                 . *|  .                |
 20        0.05126    |                 .  |* .               |
 21        0.01713    |                 .  |  .                |
 22        0.04942    |                 .  |* .               |
 23        0.00197    |                 .  |  .                |
 24       -0.01745    |                 .  |  .                |


                    Partial Autocorrelations

Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

  1        0.06897    |                 .  |* .                |
  2       -0.00830    |                 .  |  .                |
  3       -0.00041    |                 .  |  .                |
  4       -0.04877    |                 . *|  .                |
  5        0.01287    |                 .  |  .                |
  6       -0.01916    |                 .  |  .                |
  7       -0.00183    |                 .  |  .                |
  8       -0.07486    |                 . *|  .                |
  9        0.04628    |                 .  |* .                |
 10        0.11841    |                 .  |**.               |
 11        0.03697    |                 .  |* .                |
 12       -0.09207    |                 .**|  .                |
 13       -0.01457    |                 .  |  .                |
 14       -0.11485    |                 .**|  .                |
                     Cree daily log returns                    3
                           ARMA(1,1)  15:18 Friday, February 2, 2001

                      The ARIMA Procedure

                    Partial Autocorrelations

Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

 15        0.07540    |                 .  |**.               |
 16       -0.04385    |                 . *|  .                |
 17        0.00180    |                 .  |  .                |
 18       -0.16594    |                 ***|  .                |
 19        0.05041    |                 .  |* .                |
 20       -0.06240    |                 . *|  .                |
 21       -0.02732    |                 . *|  .                |
 22       -0.05643    |                 . *|  .                |
 23        0.00111    |                 .  |  .                |
```

```
  24        0.01957    |                .  |  .                |


             Autocorrelation Check for White Noise

  To     Chi-       Pr >
 Lag   Square  DF  ChiSq -------------Autocorrelations------------

   6     1.91   6 0.9276  0.069 -0.004 -0.001 -0.049  0.006 -0.017
  12    10.02  12 0.6143 -0.005 -0.072  0.034  0.125  0.052 -0.076
  18    21.95  18 0.2344 -0.030 -0.123  0.051 -0.022 -0.013 -0.157
  24    23.37  24 0.4978  0.014 -0.010 -0.037 -0.032 -0.047  0.016


              Conditional Least Squares Estimation

                              Standard                 Approx
Parameter      Estimate         Error    t Value    Pr > |t|     Lag

MU           -0.0006814     0.0045317      -0.15      0.8806       0
MA1,1          -0.18767       0.88710      -0.21      0.8326       1
AR1,1          -0.11768       0.89670      -0.13      0.8957       1


                 Constant Estimate      -0.00076
                 Variance Estimate      0.004585
                 Std Error Estimate     0.067712
                 AIC                    -638.889
                 SBC                    -628.301
                 Number of Residuals         252
          * AIC and SBC do not include log determinant.
                           Cree daily log returns                        4
                                 ARMA(1,1)  15:18 Friday, February 2, 2001

                          The ARIMA Procedure

               Correlations of Parameter Estimates

               Parameter        MU     MA1,1     AR1,1

               MU            1.000     0.005     0.006
               MA1,1         0.005     1.000     0.998
               AR1,1         0.006     0.998     1.000


                  Autocorrelation Check of Residuals

  To     Chi-       Pr >
 Lag   Square  DF  ChiSq -------------Autocorrelations------------

   6     0.75   4 0.9444  0.000  0.004  0.001 -0.049  0.010 -0.019
  12     8.54  10 0.5761  0.003 -0.075  0.032  0.118  0.050 -0.079
  18    21.12  16 0.1741 -0.014 -0.127  0.062 -0.029  0.001 -0.159
  24    22.48  22 0.4314  0.025 -0.011 -0.035 -0.026 -0.045  0.016
  30    32.65  28 0.2490  0.054  0.127  0.102 -0.023 -0.029  0.070
  36    38.16  34 0.2858 -0.055 -0.038 -0.026 -0.079  0.021  0.083
```

```
  42    47.23  40 0.2009 -0.061 -0.092 -0.004 -0.028 -0.118 -0.055
  48    49.15  46 0.3480 -0.032 -0.011 -0.004  0.027  0.054 -0.036

                    Model for variable logR

                  Estimated Mean    -0.00068

                    Autoregressive Factors

                 Factor 1:  1 + 0.11768 B**(1)

                    Moving Average Factors

                 Factor 1:  1 + 0.18767 B**(1)
```

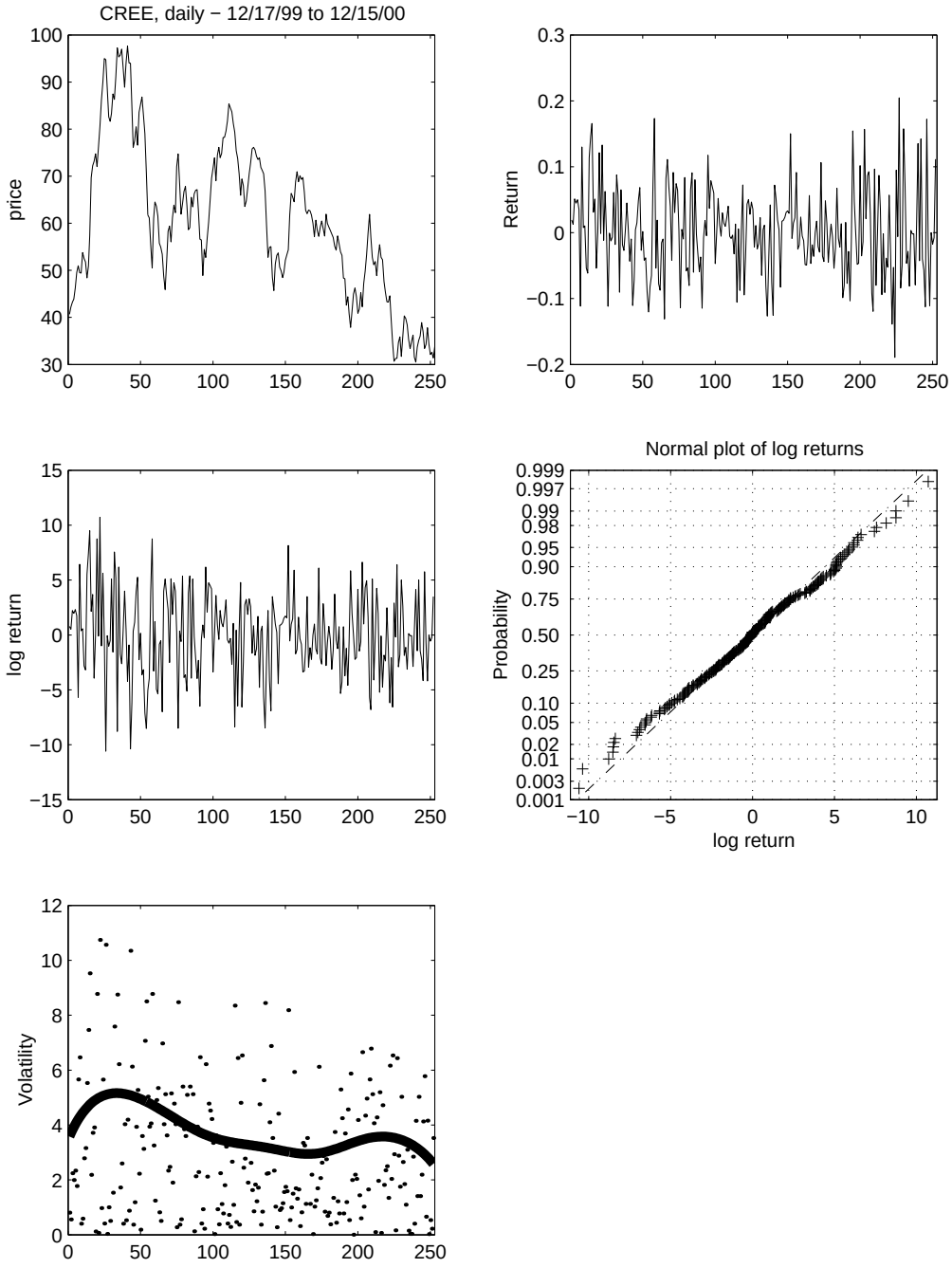


Figure 4.6: *Cree daily returns. The first plot is the prices. The second and third are the net returns and the log returns. The fourth plot is a normal probability plot of the log returns. The final plot is of the absolute log returns; there is a scatterplot smooth to help show whether the volatility is constant.*

4.8 Example: Three-month Treasury bill rates

The efficient market hypothesis predicts that log returns will be white noise, and our empirical results are that log returns have little autocorrelation even if they are not exactly white noise. Other financial time series do have substantial autocorrelation, as is shown in this example.

The time series in this example is monthly interest rates on three-month US Treasury bills from December 1950 until February 1996. The data come from Example 16.1 of Pindyck and Rubin (1998), *Econometric Models and Economic Forecasts*. The rates are plotted in Figure 4.7. The first differences look somewhat stationary, and we will fit ARMA models to the first differences.

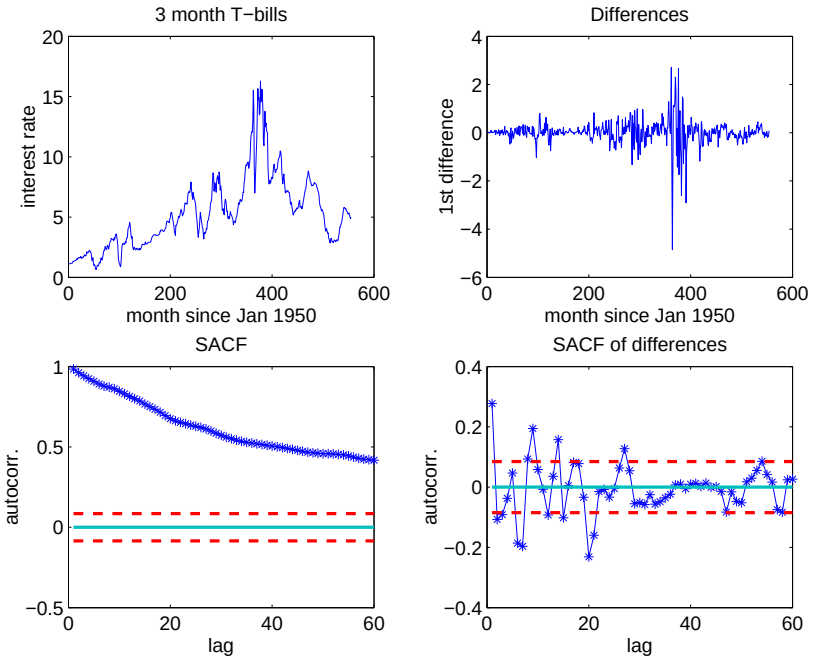


Figure 4.7: *Time series plot of 3 month Treasury bill rates, plot of first differences, and sample autocorrelation function of first differences. Monthly values from January 1950 until March 1996.*

First we try fitting an AR(10) model with ARIMA. Here is the SAS program. Note statement "identify var=z(1) ;" specifies that the model should be fit to the first differences of the variance z; z is the interest rate.

```
options linesize = 72 ;
data rate1 ;
infile 'c:\courses\or473\data\fygn.dat' ;
input date $ z;
title 'Three month treasury bills' ;
title2 'ARIMA model - to first differences' ;
proc arima ;
identify var=z(1) ;
estimate p=10 plot;
run ;
```

Here is the SAS output.

```
                    Three month treasury bills                    1
                ARIMA model - to first differences
                                  14:41 Saturday, February 3, 2001

                      The ARIMA Procedure

                      Name of Variable = z

        Period(s) of Differencing                          1
        Mean of Working Series                      0.006986
        Standard Deviation                          0.494103
        Number of Observations                           554
        Observation(s) eliminated by differencing          1


                         Autocorrelations

Lag Covariance Correlation -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

  0   0.244138     1.00000 |                    |********************|
  1   0.067690     0.27726 |                  . |******              |
  2  -0.026212     -.10736 |                 **| .                  |
  3  -0.022360     -.09159 |                 **| .                  |
  4 -0.0091143     -.03733 |                 .*| .                  |
  5   0.011399     0.04669 |                  . |*.                  |
  6  -0.045339     -.18571 |                ****| .                  |
  7  -0.047987     -.19656 |                ****| .                  |
  8   0.022734     0.09312 |                  . |**                  |
  9   0.047441     0.19432 |                  . |****                |
 10   0.014282     0.05850 |                  . |*.                  |
 11 -0.0017082     -.00700 |                  . | .                  |
 12  -0.022600     -.09257 |                 **| .                  |
```

```
13  0.0087638     0.03590 |                  . |*.                  |
14   0.038426     0.15739 |                  . |***                 |
15  -0.024885     -.10193 |                 **| .                  |
16  0.0012018     0.00492 |                  . | .                  |
17   0.020048     0.08212 |                  . |**                  |
18   0.019043     0.07800 |                  . |**                  |
19 -0.0081609     -.03343 |                 .*| .                  |
20  -0.056547     -.23162 |               *****| .                  |
21  -0.038945     -.15952 |                 ***| .                  |
22 -0.0035774     -.01465 |                  . | .                  |
23 -0.0018465     -.00756 |                  . | .                  |
24 -0.0080554     -.03300 |                  .*| .                  |

                  "." marks two standard errors


                    Three month treasury bills                    2
                ARIMA model - to first differences
                                  14:41 Saturday, February 3, 2001

                      The ARIMA Procedure

                    Inverse Autocorrelations

  Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

    1       -0.38226    |            ********| .                 |
    2        0.17388    |                  . |***                 |
    3       -0.03944    |                  .*| .                  |
    4        0.09813    |                  . |**                  |
    5       -0.15403    |                 ***| .                  |
    6        0.16052    |                  . |***                 |
    7        0.03458    |                  . |*.                  |
    8       -0.07833    |                  **| .                  |
    9       -0.01029    |                  . | .                  |
   10       -0.01264    |                  . | .                  |
   11       -0.07557    |                  **| .                  |
   12       -0.00166    |                  . | .                  |
   13        0.12786    |                  . |***                 |
   14       -0.22060    |                ****| .                  |
   15        0.19060    |                  . |****                |
   16       -0.10958    |                  **| .                  |
   17        0.03736    |                  . |*.                  |
   18       -0.05356    |                  .*| .                  |
   19        0.07262    |                  . |*.                  |
   20        0.03663    |                  . |*.                  |
   21        0.03580    |                  . |*.                  |
   22        0.02890    |                  . |*.                  |
   23        0.00507    |                  . | .                  |
   24        0.00765    |                  . | .                  |


                    Partial Autocorrelations

  Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
```

```
 1        0.27726    |                  . |******              |
 2       -0.19958    |                ****| .                  |
 3       -0.00061    |                  . | .                  |
 4       -0.03172    |                  .*| .                  |
 5        0.05661    |                  . |*.                  |
 6       -0.25850    |               *****| .                  |
 7       -0.05221    |                  .*| .                  |
 8        0.14071    |                  . |***                 |
 9        0.08439    |                  . |**                  |
10       -0.04699    |                  .*| .                  |
11        0.06148    |                  . |*.                  |
12       -0.11389    |                  **| .                  |
13        0.05561    |                  . |*.                  |
14        0.13716    |                  . |***                 |
                  Three month treasury bills                     3
               ARIMA model - to first differences
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                     The ARIMA Procedure

                   Partial Autocorrelations

 Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

 15       -0.13273    |                ***| .                  |
 16        0.15741    |                  . |***                 |
 17        0.02301    |                  . | .                  |
 18        0.01777    |                  . | .                  |
 19       -0.13330    |                ***| .                  |
 20       -0.08447    |                 **| .                  |
 21       -0.07718    |                 **| .                  |
 22       -0.04553    |                 .*| .                  |
 23       -0.01479    |                  . | .                  |
 24       -0.01071    |                  . | .                  |


             Autocorrelation Check for White Noise

  To     Chi-       Pr >
 Lag   Square  DF  ChiSq ------------Autocorrelations-----------

   6    75.33   6 <.0001  0.277 -0.107 -0.092 -0.037  0.047 -0.186
  12   130.15  12 <.0001 -0.197  0.093  0.194  0.059 -0.007 -0.093
  18   158.33  18 <.0001  0.036  0.157 -0.102  0.005  0.082  0.078
  24   205.42  24 <.0001 -0.033 -0.232 -0.160 -0.015 -0.008 -0.033


              Conditional Least Squares Estimation

                              Standard                 Approx
Parameter      Estimate         Error    t Value    Pr > |t|     Lag

MU           0.0071463       0.02056       0.35      0.7283       0
AR1,1          0.33494       0.04287       7.81      <.0001       1
AR1,2         -0.16456       0.04501      -3.66      0.0003       2
AR1,3          0.01712       0.04535       0.38      0.7060       3
```

AR1,4	-0.10901	0.04522	-2.41	0.0163	4
AR1,5	0.14252	0.04451	3.20	0.0014	5
AR1,6	-0.21560	0.04451	-4.84	<.0001	6
AR1,7	-0.08347	0.04522	-1.85	0.0655	7
AR1,8	0.10382	0.04536	2.29	0.0225	8
AR1,9	0.10007	0.04502	2.22	0.0267	9
AR1,10	-0.04723	0.04290	-1.10	0.2714	10

Constant Estimate	0.006585
Variance Estimate	0.198648
Std Error Estimate	0.445699

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The ARIMA Procedure

AIC	687.6855
SBC	735.1743
Number of Residuals	554

* AIC and SBC do not include log determinant.

Correlations of Parameter Estimates

Parameter	MU	AR1,1	AR1,2	AR1,3	AR1,4	AR1,5
MU	1.000	0.001	-0.000	-0.001	-0.001	-0.000
AR1,1	0.001	1.000	-0.315	0.160	-0.020	0.095
AR1,2	-0.000	-0.315	1.000	-0.357	0.166	-0.033
AR1,3	-0.001	0.160	-0.357	1.000	-0.350	0.204
AR1,4	-0.001	-0.020	0.166	-0.350	1.000	-0.375
AR1,5	-0.000	0.095	-0.033	0.204	-0.375	1.000
AR1,6	-0.001	-0.131	0.122	-0.068	0.218	-0.367
AR1,7	-0.001	0.200	-0.178	0.161	-0.078	0.218
AR1,8	-0.001	0.080	0.163	-0.166	0.161	-0.068
AR1,9	-0.001	-0.106	0.123	0.163	-0.178	0.122
AR1,10	-0.003	-0.085	-0.106	0.080	0.200	-0.131

Correlations of Parameter Estimates

Parameter	AR1,6	AR1,7	AR1,8	AR1,9	AR1,10
MU	-0.001	-0.001	-0.001	-0.001	-0.003
AR1,1	-0.131	0.200	0.080	-0.106	-0.085
AR1,2	0.122	-0.178	0.163	0.123	-0.106
AR1,3	-0.068	0.161	-0.166	0.163	0.080
AR1,4	0.218	-0.078	0.161	-0.178	0.200
AR1,5	-0.367	0.218	-0.068	0.122	-0.131
AR1,6	1.000	-0.375	0.204	-0.033	0.096
AR1,7	-0.375	1.000	-0.350	0.166	-0.020
AR1,8	0.204	-0.350	1.000	-0.357	0.161
AR1,9	-0.033	0.166	-0.357	1.000	-0.315
AR1,10	0.096	-0.020	0.161	-0.315	1.000

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ARIMA model - to first differences
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The ARIMA Procedure

Autocorrelation Check of Residuals

To Lag	Chi-Square	DF	Pr > ChiSq	Autocorrelations					
6	0.00	0	<.0001	0.003	-0.011	0.003	0.021	-0.015	-0.031
12	9.56	2	0.0084	0.036	-0.001	-0.031	0.018	0.105	-0.040
18	42.72	8	<.0001	-0.076	0.177	-0.115	0.081	0.019	0.025
24	62.06	14	<.0001	-0.062	-0.149	-0.078	-0.025	-0.024	-0.013
30	65.76	20	<.0001	0.002	0.008	0.045	0.048	-0.043	-0.007
36	73.52	26	<.0001	-0.070	-0.004	-0.051	-0.003	-0.053	-0.052
42	74.14	32	<.0001	-0.007	0.028	-0.007	-0.005	0.010	0.006
48	82.20	38	<.0001	-0.011	-0.000	-0.006	0.001	-0.103	0.050

Autocorrelation Plot of Residuals

Lag	Covariance	Correlation	-1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
0	0.198648	1.00000	|********************|
1	0.00057812	0.00291	. | . |
2	-0.0020959	-.01055	. | . |
3	0.00068451	0.00345	. | . |
4	0.0041792	0.02104	. | . |
5	-0.0030362	-.01528	. | . |
6	-0.0061377	-.03090	.*| . |
7	0.0071315	0.03590	. |*. |
8	-0.0001693	-.00085	. | . |
9	-0.0061781	-.03110	.*| . |
10	0.0036055	0.01815	. | . |
11	0.020788	0.10465	. |** |
12	-0.0078818	-.03968	.*| . |
13	-0.015171	-.07637	**| . |
14	0.035240	0.17740	. |**** |
15	-0.022934	-.11545	**| . |
16	0.016000	0.08054	. |** |
17	0.0037288	0.01877	. | . |
18	0.0049781	0.02506	. |*. |
19	-0.012221	-.06152	.*| . |
20	-0.029590	-.14896	***| . |
21	-0.015566	-.07836	**| . |
22	-0.0050098	-.02522	.*| . |
23	-0.0048445	-.02439	. | . |
24	-0.0026174	-.01318	. | . |

"." marks two standard errors

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ARIMA model - to first differences
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The ARIMA Procedure

Inverse Autocorrelations

Lag	Correlation	-1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
1	-0.04462	.*| .
2	0.02988	. |*.
3	0.02921	. |*.
4	-0.04817	.*| .
5	0.00308	. | .
6	0.02072	. | .
7	-0.02134	. | .
8	-0.01272	. | .
9	0.01308	. | .
10	-0.02753	.*| .
11	-0.10241	**| .
12	0.03617	. |*.
13	0.06350	. |*.
14	-0.16306	***| .
15	0.12298	. |**
16	-0.08990	**| .
17	-0.02141	. | .
18	-0.00130	. | .
19	0.04419	. |*.
20	0.11901	. |**
21	0.08929	. |**
22	0.02613	. |*.
23	0.00628	. | .
24	0.00879	. | .

Partial Autocorrelations

Lag	Correlation	-1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
1	0.00291	. | .
2	-0.01056	. | .
3	0.00351	. | .
4	0.02091	. | .
5	-0.01534	. | .
6	-0.03040	.*| .
7	0.03569	. |*.
8	-0.00204	. | .
9	-0.02966	.*| .
10	0.01926	. | .
11	0.10200	. |**
12	-0.04035	.*| .
13	-0.07248	.*| .
14	0.17834	. |****

The ARIMA Procedure

Partial Autocorrelations

Lag	Correlation	-1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
15	-0.13109	***| . |
16	0.09936	. |** |
17	0.02268	. | . |
18	0.00293	. | . |
19	-0.05597	.*| . |
20	-0.13881	***| . |
21	-0.10044	**| . |
22	-0.02905	.*| . |
23	-0.00750	. | . |
24	-0.00979	. | . |

Model for variable z

Estimated Mean	0.007146
Period(s) of Differencing	1

Autoregressive Factors

Factor 1: 1 - 0.33494 B**(1) + 0.16456 B**(2) - 0.01712 B**(3) + 0.10901 B**(4) - 0.14252 B**(5) + 0.2156 B**(6) + 0.08347 B**(7) - 0.10382 B**(8) - 0.10007 B**(9) + 0.04723 B**(10)

The AR(10) model does not fit well. Next we try an AR(24) model with backfitting. Here is the SAS program:

```
options linesize = 72 ;
data rate1 ;
infile 'c:\courses\or473\data\fygn.dat' ;
input date $ z;
zdif=dif(z) ;
title 'Three month treasury bills' ;
title2 'AR(24) model to first differences with backfitting' ;

proc autoreg ;
model zdif= / nlag=24 backstep;
run ;
```

Here is the output.

```
                Three month treasury bills                      1
        AR(24) model to first differences with backfitting
                              10:32 Wednesday, February 14, 2001

                    The AUTOREG Procedure

                   Dependent Variable    zdif


                Ordinary Least Squares Estimates

    SSE                 135.25253    DFE                    553
    MSE                   0.24458    Root MSE           0.49455
    SBC                 797.34939    AIC             793.032225
    Regress R-Square        0.0000   Total R-Square       0.0000
    Durbin-Watson           1.4454


                                    Standard                 Approx
  Variable        DF     Estimate        Error    t Value    Pr > |t|

  Intercept        1     0.006986       0.0210       0.33      0.7397


                  Estimates of Autocorrelations

                  Lag    Covariance     Correlation

                    0        0.2441        1.000000
                    1        0.0677        0.277260
                    2       -0.0262       -0.107364
                    3       -0.0224       -0.091587
                    4      -0.00911       -0.037332
```

5	0.0114	0.046690
6	-0.0453	-0.185710
7	-0.0480	-0.196558
8	0.0227	0.093118
9	0.0474	0.194318

Estimates of Autocorrelations

```
Lag    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

  0    |                    |********************|
  1    |                    |******              |
  2    |                  **|                    |
  3    |                  **|                    |
  4    |                   *|                    |
  5    |                    |*                   |
  6    |                ****|                    |
  7    |                ****|                    |
  8    |                    |**                  |
  9    |                    |****                |
```

Three month treasury bills 2

AR(24) model to first differences with backfitting

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The AUTOREG Procedure

Estimates of Autocorrelations

Lag	Covariance	Correlation
10	0.0143	0.058501
11	-0.00171	-0.006997
12	-0.0226	-0.092572
13	0.00876	0.035897
14	0.0384	0.157393
15	-0.0249	-0.101930
16	0.00120	0.004923
17	0.0200	0.082117
18	0.0190	0.078001
19	-0.00816	-0.033427
20	-0.0565	-0.231618
21	-0.0389	-0.159520
22	-0.00358	-0.014653
23	-0.00185	-0.007563
24	-0.00806	-0.032995

Estimates of Autocorrelations

```
Lag    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

 10    |                    |*                   |
 11    |                    |                    |
 12    |                  **|                    |
 13    |                    |*                   |
 14    |                    |***                 |
 15    |                  **|                    |
```

```
16    |                    |                    |
17    |                    |**                  |
18    |                    |**                  |
19    |                   *|                    |
20    |                *****|                    |
21    |                  ***|                    |
22    |                     |                    |
23    |                     |                    |
24    |                   *|                    |
```


The AUTOREG Procedure

Backward Elimination of
Autoregressive Terms

Lag	Estimate	t Value	Pr > |t|
10	0.007567	0.16	0.8721
23	0.010212	0.22	0.8241
17	0.008951	0.19	0.8492
3	-0.014390	-0.32	0.7496
24	0.015798	0.40	0.6907
13	0.041434	0.92	0.3605
7	0.038880	0.85	0.3964
18	-0.037456	-0.90	0.3702
22	0.042555	1.02	0.3090
20	0.058230	1.31	0.1912
4	0.059903	1.48	0.1389
9	-0.058141	-1.42	0.1562

Preliminary MSE 0.1765

Estimates of Autoregressive Parameters

Lag	Coefficient	Standard Error	t Value
1	-0.388246	0.040419	-9.61
2	0.200242	0.040438	4.95
5	-0.108069	0.040513	-2.67
6	0.249095	0.039719	6.27
8	-0.103462	0.039668	-2.61
11	-0.102896	0.040278	-2.55
12	0.119950	0.040704	2.95
14	-0.204702	0.040427	-5.06
15	0.223381	0.042441	5.26
16	-0.151917	0.040811	-3.72
19	0.103356	0.038847	2.66
21	0.108074	0.039511	2.74

```
                Three month treasury bills                      4
          AR(24) model to first differences with backfitting
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                     The AUTOREG Procedure

                          Expected
                        Autocorrelations

                        Lag     Autocorr

                          0       1.0000
                          1       0.2840
                          2      -0.1196
                          3      -0.0801
                          4       0.0273
                          5       0.0656
                          6      -0.1914
                          7      -0.1923
                          8       0.0880
                          9       0.1549
                         10       0.0223
                         11      -0.0229
                         12      -0.0737
                         13       0.0767
                         14       0.1628
                         15      -0.1000
                         16      -0.0017
                         17       0.0685
                         18       0.0437
                         19      -0.0638
                         20      -0.1968
                         21      -0.1296


                     Yule-Walker Estimates

   SSE                 97.7597462    DFE                     541
   MSE                    0.18070    Root MSE            0.42509
   SBC                695.767655    AIC                639.644514
   Regress R-Square        0.0000    Total R-Square        0.2772
   Durbin-Watson           2.0627


                                    Standard                  Approx
Variable        DF     Estimate        Error    t Value    Pr > |t|

Intercept        1     0.006664       0.0192       0.35      0.7289
                Three month treasury bills                      5
          AR(24) model to first differences with backfitting
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                     The AUTOREG Procedure
```

```
    Expected
Autocorrelations

Lag     Autocorr

  0       1.0000
  1       0.2840
  2      -0.1196
  3      -0.0801
  4       0.0273
  5       0.0656
  6      -0.1914
  7      -0.1923
  8       0.0880
  9       0.1549
 10       0.0223
 11      -0.0229
 12      -0.0737
 13       0.0767
 14       0.1628
 15      -0.1000
 16      -0.0017
 17       0.0685
 18       0.0437
 19      -0.0638
 20      -0.1968
 21      -0.1296
```

4.9 Forecasting

ARIMA models are often used to forecast future values of a time series. Consider forecasting using an AR(1) process. Suppose that we have data $y_1, \ldots, y_n$ and estimates $\widehat{\mu}$ and $\widehat{\phi}$. Then we estimate y_{n+1} by

$$\widehat{y}_{n+1} := \widehat{\mu} + \widehat{\phi}(y_n - \widehat{\mu})$$

and y_{t+2} by

$$\widehat{y}_{n+2} := \widehat{\mu} + \widehat{\phi}(\widehat{y}_{n+1} - \widehat{\mu}) = \widehat{\phi}\{\widehat{\phi}(y_n - \widehat{\mu})\},$$

etc. In general, $\widehat{y}_{n+k} = \widehat{\mu} + \widehat{\phi}^k(y_n - \widehat{\mu})$. If $\widehat{\phi} < 1$ as is expected for a stationary series, then as k increases the forecasts will decay exponentially fast to $\widehat{\mu}$.

Forecasting general AR(p) processes is similar. For example, for an AR(2) process

$$\widehat{y}_{n+1} := \widehat{\mu} + \widehat{\phi}_1(y_n - \widehat{\mu}) + \widehat{\phi}_2(y_{n-1} - \widehat{\mu})$$

and

$$\widehat{y}_{n+2} := \widehat{\mu} + \widehat{\phi}_1(\widehat{y}_{n+1} - \widehat{\mu}) + \widehat{\phi}_2(y_n - \widehat{\mu}).$$

Forecasting ARMA and ARIMA processes is only slightly more complicated than forecasting AR processes and is discussed in time series courses such as ORIE 563. Moreover, the forecasts can be generated automatically by statistical software such as MINITAB and SAS, so you don't need to know the details in order to forecast.

4.9.1 GE daily returns

We have learned that fitting an ARIMA(1,0,0) model to log returns is equivalent to fitting an ARIMA(1,1,0) model to the log prices. Here we will fit both models to the GE daily price data. Figure 4.8 shows the forecasts of the log returns up to 24 days ahead. The forecasts are given in red and 95% confidence limits on the forecasts are show in blue. The observed time series is plotted in black.

Next we fit an ARIMA(1,1,0) model to the log prices. Although this model is equivalent to the last model, it generates forecasts of the log prices, not the log returns. (MINITAB always forecasts the input series.) The forecasts are given in Figure 4.9. Notice that the forecasts predict that the price of GE will stay constant, but the confidences limits on the forecasts get wider as we forecast further ahead. This is exactly the type of behavior we would expect from a random walk [ARIMA(0,1,0)] model. The ARIMA(1,1,0) model for the log prices isn't quite a random walk model, but it is similar to a random walk model with zero drift ($\mu = 0$) since $\widehat{\phi}$ is close to 0 and $\widehat{\mu}$ is *extremely* close to 0.

The forecast limits suggest that accurately forecasting GE stock prices far into the future is pretty hopeless. For practical purposes the log prices behave like a random walk so that the prices behave like a geometric random walk.

Figures 4.8 and 4.9 show that forecasts of a stationary process behave very differently from forecasts of a nonstationary process. The forecasts of the stationary AR(1) process in Figures 4.8 converge to the mean of the observed series and the distance between the confidence limits converges to a constant muliple of $\widehat{\sigma}$. In contrast, the forecasts of the nonstationary ARIMA(1,1,0) process in Figure 4.9 are close to the average of the last few observations rather than equaling the average of all the observations. Also,

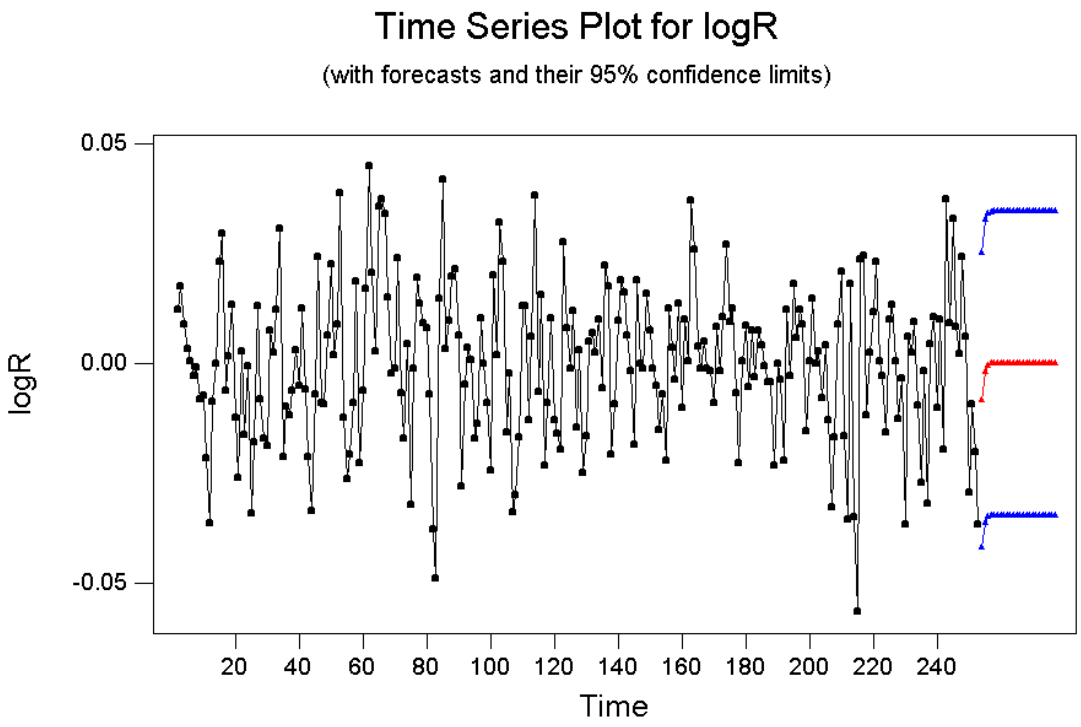


Figure 4.8: *Time series plot of the daily GE log returns with forecasts from an AR(1) model.*

for the nonstationary process the distance between the confidence limits increases as one forecasts farther into the future.

4.10 Summary

- time series = sequence of observations
- a process is *weakly stationary* if its mean, variance, and covariance are unchanged by time shifts
- thus $X_1, X_2, \ldots$ is a weakly stationary process if
  - $E(X_i) = \mu$ (a constant) for all i

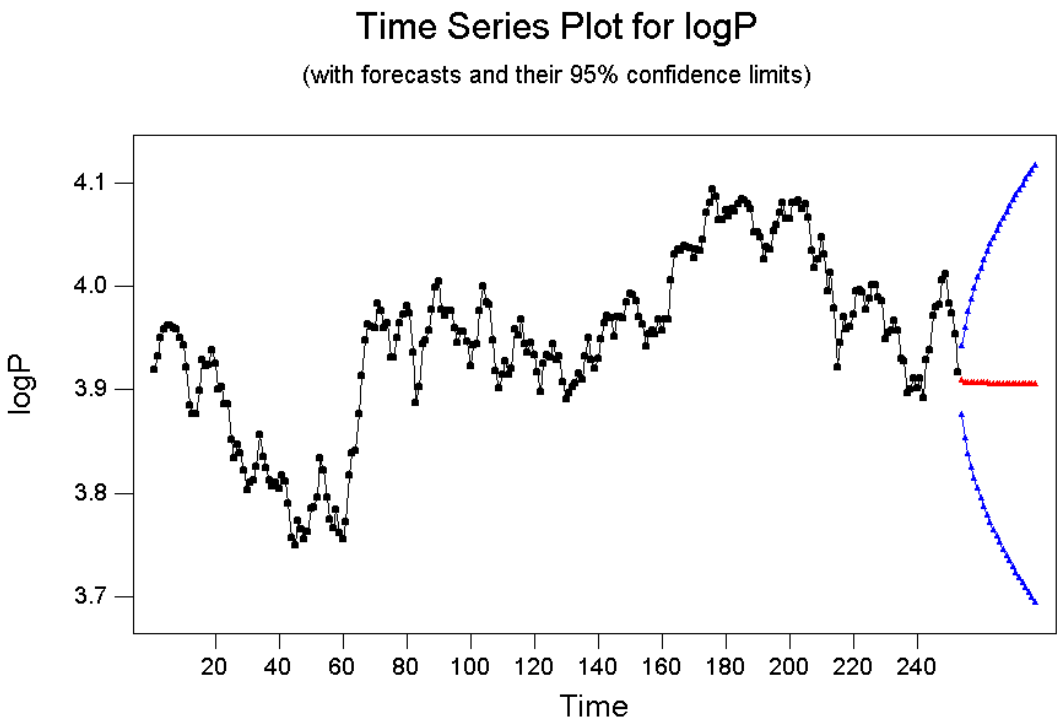


Figure 4.9: *Time series plot of the daily GE log prices with forecasts from an ARIMA(1,1,0) model.*

  - Var(X_i) $= \sigma^2$ (a constant) for all i
  - Corr(X_i, X_j) $= \rho(|i-j|)$ for all i and j for some function ρ
- covariance between X_t and X_{t+h} is denoted by $\gamma(h)$
  - $\gamma(\cdot)$ is called the autocovariance function
  - Note that $\gamma(h) = \sigma^2\rho(h)$ and that $\gamma(0) = \sigma^2$ since $\rho(0) = 1$
- therefore $\rho(h) = \gamma(h)/\sigma^2$
- $X_1, X_2, \ldots$ is WN(0,σ^2) if
  - $E(X_i) = 0$ for all i
  - Var(X_i) $= \sigma^2$ (a constant) for all i

  - Corr$(X_i, X_j) = 0$ for all $i \neq j$

- AR(1) is simplest AR process
- $\epsilon_1, \epsilon_2, \ldots$ are WH(0,σ_ϵ^2)
- $y_1, y_2, \ldots$ is an AR(1) process if

$$y_t - \mu = \phi(y_{t-1} - \mu) + \epsilon_t \tag{4.7}$$

  for all t

- If $|\phi| < 1$, then $y_1, \ldots$ is a weakly stationary process
- mean is μ
-

$$\rho(h) = \text{Corr}(y_t, y_{t+h}) = \phi^{|h|} \quad \forall t$$

- The least squares estimation of ϕ and μ minimize

$$\sum_{t=2}^{n} \Big[\{y_t - \mu\} - \{\phi(y_{t-1} - \mu)\}\Big]^2$$

  - both MINITAB or SAS have special procedures for fitting AR models

- **Residuals**

$$\widehat{\epsilon}_t = y_t - \widehat{\mu} - \widehat{\phi}(y_{t-1} - \widehat{\mu})$$

- autocorrelation in residuals evidence against AR(1) assumption
- y_t is AR(p) process if

$$(y_t - \mu) = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \epsilon_t$$

- backwards operator B is defined by

$$B\, y_t = y_{t-1}$$

- ARMA(p, q) process satisfies the equation

$$(1 - \phi\, B - \cdots - \phi_p\, B^p)(y_t - \mu) = (1 - \theta_1\, B - \ldots - \theta_q\, B^q)\epsilon_t \tag{4.8}$$

- differencing operator is $\Delta = 1 - B$ so that

$$\Delta y_t = y_t - B\, y_t = y_t - y_{t-1}$$

- a time series y_t is said to be ARIMA(p, d, q) if $\Delta^d y_t$ is ARMA(p, q)
- AIC and SBC are model selection criteria based on the log-likelihood
- "best" model by either criterion is the model that minimizes that criterion
- therefore, AIC and SBC try to tradeoff
  - good fit to the data measured by L
  - the desire to use few parameters
- SBC penalizes $p + q$ more than AIC
- ARIMA models can forecast future values
- consider forecasting using an AR(1) process
- we estimate y_{n+1} by

$$\widehat{y}_{n+1} := \widehat{\mu} + \widehat{\phi}(y_n - \widehat{\mu})$$

- and y_{n+2} by

$$\widehat{y}_{n+2} := \widehat{\mu} + \widehat{\phi}(\widehat{y}_{n+1} - \widehat{\mu}) = \widehat{\phi}\{\widehat{\phi}(y_n - \widehat{\mu})\},$$

  etc.
- forecasting ARMA and ARIMA processes is slightly more complicated
- is discussed in time series courses such as ORIE 563
  - the forecasts can be generated automatically by statistical software such as MINITAB and SAS

4.11 References

Enders, W. (1995), *Applied Econometric Time Series*, Wiley, New York. [Good introduction to time series analysis of economic data.]

SAS Institute (1993), *SAS/ETS User's Guide, Version 6, 2nd Edition*, SAS Institute, Cary, NC. [Discusses the time series software available in SAS.]

Chapter 5

Portfolio Selection: 03/29/02

5.1 Trading off expected return and risk

How should we invest our wealth? There are two principles:

- we want to maximize the expected return
- we want to minimize the risk = variance of return

These goals are somewhat at odds because riskier assets generally have a higher expected return, since investors demand a reward for bearing risk. The difference between the expected return of a risky asset and the risk free rate of return is called the risk premium. Without risk premiums, no one would invest in risky assets.

Nonetheless, there are optimal compromises between expected return and risk. In this chapter we will see how to maximize expected return subject to an upper bound on the risk, or to minimize the risk subject to a lower bound on the expected return.

The key concept that we will discuss is reduction of risk by diversifying the portfolio of assets held. Diversification was not always considered as favorably as it is now.

The investment philosophy of Keynes

The famous British economist, John Maynard Keynes, did not believe in diversifying a portfolio. He wrote:

> ... the management of stock exchange investment of any kind is a low pursuit ... from which it is a good thing for most members of society to be free

> I am in favor of having as large a unit as market conditions will allow ... to suppose that safety-first consists in having a small gamble in a large number of different [companies] where I have no information to reach a good judgement, as compared with a substantial stake in a company where ones's information is adequate, strikes me as a travesty of investment policy

This quote is taken from Bernstein, *Capital Ideas: The Improbable Origins of Modern Wall Street*.

Keynes is advocating stock picking or "fundamental analysis." But the semi-strong version of the EMH says that fundamental analysis does not lead to economic profit. Of course, Keynes lived well before the EMH and one wonders what Keynes would think about diversification if he were alive now. Modern portfolio theory takes a very different viewpoint than Keynes. This is not to say that Keynes was wrong. Keynes was investing on a long time horizon, and fundamental analysis, if done well, might be very successful in the long run. However, portfolio managers are judged on short-term successes. Also, using fundamental analysis to find bargains is probably more difficult now than in Keynes's time.

5.2 One risky asset and one risk-free asset

We will start with a simple example where we have

- one risky asset, which could be a portfolio, e.g., a mutual fund
  - expected return is .15
  - standard deviation of the return is .25
- one risk-free asset, e.g., a 30-day T-bill
  - expected value of the return is .06
  - standard deviation of the return is 0 by definition of "risk-free."

We are faced with the problem of constructing an investment portfolio that we will hold for one time period which could be an hour, a day, a month, a quarter, a year, ten years, etc. At the end of the time period we might want to readjust the portfolio, so for now we are only looking at returns over one time period. Suppose that

- a fraction w of our wealth is invested in the risky asset

- the remaining fraction $1 - w$ is invested in the risk-free asset
- then the expected return is $E(R) = w(.15) + (1-w)(.06) = .06 + .09w$.
- the variance of the return is

$$\sigma_R^2 = w^2 (.25)^2 + (1-w)^2 (0)^2 = w^2 (.25)^2.$$

  and the standard deviation of the return is $\sigma_R = .25\, w$.

Would $w > 1$ make any sense?

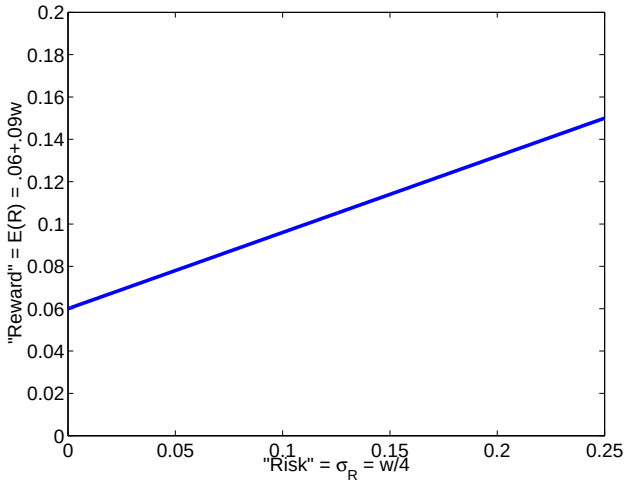


Figure 5.1: *Expected return for a portfolio with allocation w to the risky asset with expected return 0.15 and allocation $1 - w$ to the risk-free return with return 0.06.*

To decide what proportion w of one's wealth to invest in the risky asset one either chooses the expected return $E(R)$ one wants or the amount of risk σ_R one is willing to live with. Once either $E(R)$ or σ_R is chosen, w can be determined.

Question: Suppose you want an expected return of .10? What should w be? [Answer: 4/9]

Question: Suppose you want $\sigma_R = .05$. What should w be? [Answer: 0.2]

More generally, if the expected returns on the risky and risk-free assets are μ_1 and μ_f and if the standard deviation of the risky asset is σ_1, then

the expected return on the portfolio is $w\mu_1 + (1-w)\mu_f$ while the standard deviation of the portfolio's return is $w\,\sigma_1$.

This model is simple but not as useless as it might seem at first. As will be discussed later in this course, finding an optimal portfolio can be achieved in two steps.

1. finding the "optimal" portfolio of risky assets, called the "tangency portfolio"
2. finding the appropriate mix of the risk-free asset and the tangency portfolio from step one

So we now know how to do the second step. What we need to learn is how to mix optimally a number of risky assets; we will do that in the next sections. First, we look at a related example.

5.2.1 Example

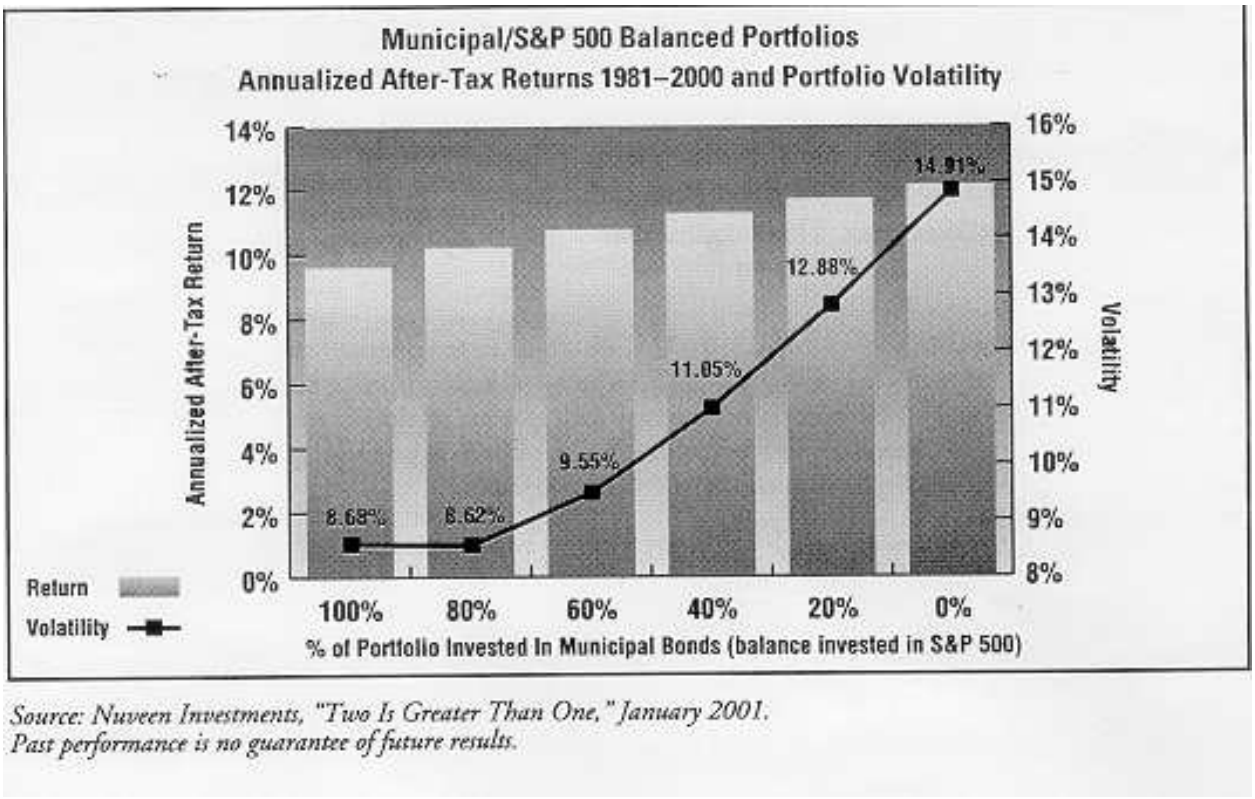
In the February 2001 issue of PaineWebber's *Investment Intelligence: A Report for Our Clients*, the advantages of holding municipal bonds are touted. PaineWebber says "The chart at the right shows that a 20% municipal/80% S&P 500 mix sacrificed only 0.42% annual after-tax return relative to a 100% S&P 500 portfolio, while reducing risk by 13.6% from 14.91% to 12.88%. The chart is show here as Figure 5.2. Although PaineWebber's point is correct, the chart is cleverly designed to over-emphasize the reduction in volatility; how?

5.2.2 Estimating $E(R)$ and σ_R

The risk-free rate, μ_f, will be known since Treasury bill rates are published in most newspapers and web sites providing financial information.

What should we use as the values of $E(R)$ and σ_R? If returns on the asset are assumed to be stationary, then we can take a time series of past returns and use the sample mean and standard deviation. Whether the stationarity assumption is realistic or not is always debatable. If we think that $E(R)$ and σ_R will be different than in the past, we could subjectively adjust these estimates upward or downward according to our opinions, but we must live with the consequences if our opinions prove to be incorrect.

Another question is how long a time series to use, that is how far back in time when should gather data. A long series, say 10 or 20 years, will



Source: Nuveen Investments, "Two Is Greater Than One," January 2001.
Past performance is no guarantee of future results.

Figure 5.2: *Chart from PaineWebber newsletter showing reduction in volatility by mixing municipal bonds with the S&P 500 index.*

give much less variable estimates. However, if the series is not stationary but rather has slowly drifting parameters, then a shorter series (maybe 1 or 2 years) will be more representative of the future. Almost every time series of returns is nearly stationary over short enough time periods.

5.3 Two risky assets

5.3.1 Risk versus expected return

The mathematics of mixing risky assets is most easily understood when there are only two risky assets. This is where we will start.

Suppose the two risky assets have returns R_1 and R_2 and that we mix them in proportions w and $1 - w$, respectively. The return is $R = wR_1 +$

$(1-w)R_2$. The expected return on the portfolio is $E(R) = w\mu_1 + (1-w)\mu_2$. Let ρ_{12} be the correlation between the returns on the two risky assets. The variance of the return on the portfolio is

$$\sigma_R^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\,\sigma_1\sigma_2.$$

[Note: $\sigma_{R_1,R_2} = \rho_{12}\sigma_1\sigma_2$.]

Example:

If $\mu_1 = .14$, $\mu_2 = .08$, $\sigma_1 = .2$, $\sigma_2 = .15$, and $\rho_{12} = 0$, then

$$E(R) = .08 + .06w.$$

Also, because $\rho_{12} = 0$ in this example

$$\sigma_R^2 = (.2)^2\,w^2 + (.15)^2\,(1-w)^2.$$

Using differential calculus, one can easily show that the portfolio with the minimum risk is $w = .045/.125 = .36$. For this portfolio $E(R) = .08 + (.06)(.36) = .1016$ and $\sigma_R = \sqrt{(.2)^2(.36)^2 + (.15)^2(.64)^2} = .12$.

Here are values of $E(R)$ and σ_R for some other values of w:

w	$E(R)$	σ_R
0	.080	.150
1/4	.095	.123
1/2	.110	.125
3/4	.125	.155
1	.140	.200

The somewhat parabolic curve[1] in Figure 5.3 is the locus of values of $(\sigma_R, E(R))$ when $0 \le w \le 1$. This locus is called the *efficient frontier* for reasons that will be discussed later. The points labeled R_1 and R_2 corresponds to $w = 1$ and $w = 0$, respectively. The other features of this figure will be explained in the Section 5.4.

5.3.2 Estimating means, standard deviations, and covariances

Estimates of μ_1 and σ_1 can be obtained from a univariate times series of past returns on the first risky asset; denote this time series by $R_{1,1}, \ldots, R_{1,n}$ where the first subscript indicates the asset and the second subscript is for

[1] In fact, the curve would be parabolic if σ_R^2 were plotted on the x-axis instead of σ_R. Why?

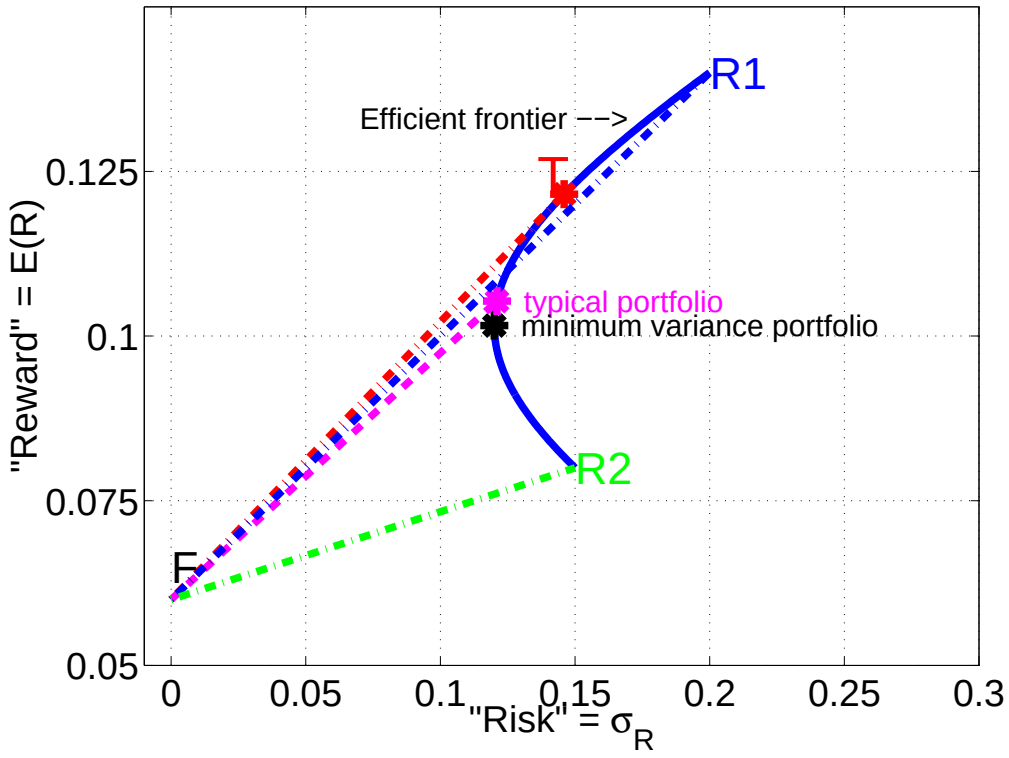


Figure 5.3: *Expected return versus risk. The somewhat parabolic curve is the locus of portfolios combining the two risky assets and is called the efficient frontier. The lines are the locus of portfolios of two risky assets and the risk-free asset. F = risk-free asset. T = tangency portfolio. R_1 is the first risky asset. R_2 is the second risky asset.*

time. Let $\overline{R}_1$ and s_{R_1} be the sample mean and standard deviation of this series. Similarly, μ_2 and σ_2 can be estimated from a time series of past returns on the second risky asset. The covariance σ_{12} can be estimated by sample covariance

$$\widehat{\sigma}_{12} = n^{-1} \sum_{t=1}^{n} (R_{1,t} - \overline{R}_1)(R_{2,t} - \overline{R}_2).$$

The correlation ρ_{12} can be estimated by the sample correlation

$$\widehat{\rho}_{12} = \frac{\widehat{\sigma}_{12}}{s_1 s_2}.$$

$\widehat{\rho}_{12}$, sometimes denoted by r_{12}, is called the cross-correlation coefficient between R_1 and R_2 at lag 0, since we are correlating the return on the first risky asset with the return on the second during the same time periods. Cross-correlations at other lags can be defined but are not needed here. In fact, we can define a cross-correlation function, which is a function of lag. The cross-correlation function plays an important role in the analysis of multivariate time series.

Sample correlations and covariances can be computed on MINITAB. Go to "Stat," then "Basic statistics," and then "Correlation" or "Covariance."

5.4 Combining two risky assets with a risk-free asset

5.4.1 Tangency portfolio with two risky assets

As mentioned in Section 5.3.1 each point on the efficient frontier in Figure 5.3 is $(\sigma_R, E(R))$ for some value of w between 0 and 1. If we fix w, then we have a fixed portfolio of the two risky assets. Now let us mix that portfolio of risky assets with the risk-free asset. The point F in Figure 5.3 gives $(\sigma_R, E(R))$ for the risk-free asset; of course $\sigma_R = 0$ at F. The possible values of $(\sigma_R, E(R))$ for a portfolio consisting of the fixed portfolio of two risky assets and the risk-free asset is a line connecting the point F with a point on the efficient frontier, e.g., the dashed line. The dotted line connecting F with R_1 mixes the risk-free asset with the first risky asset.

Notice that the dotted line lies above the dashed line. This means that for any value of σ_R, the dotted line gives a higher expected return than the dashed line. The slope of any line is called the "Sharpe ratio" of the line; it is named after William Sharpe whom we have met before in Section 3.8 and will meet again in Chapter 7. Sharpe's ratio can be thought of as a "reward-to-risk" ratio. It is the ratio of the "excess expected return" to the risk as measured by the standard deviation.

Clearly, the bigger the Sharpe ratio the better. Why? The point T on the parabola represents the portfolio with the highest Sharpe ratio. It is the optimal portfolio for the purpose of mixing with the risk-free asset. This portfolio is called the "tangency portfolio" since its line is tangent to the efficient frontier.

Key result: The optimal or "efficient" portfolios mix the tangency portfolio of two risky assets with the risk-free asset. Each efficient portfolio has two properties:

- it has a higher expected return than any other portfolio with the same (or smaller) risk
- it has a smaller risk than any other portfolio with the same (or smaller) expected return.

Thus we can only improve (reduce) the risk of an efficient portfolio by accepted a worse (smaller) expected return, and we can only improve (increase) the expected return of an efficient portfolio by accepting worse (higher) risk.

Note that all efficient portfolios use the same mix of the two risky assets, namely the tangency portfolio. Only the proportion allocated to the tangency portfolio and the proportion allocated to the risk-free asset vary.

5.4.2 Finding the tangency portfolio

Given the importance of the tangency portfolio, you may be wondering "how do we find it?"

Again let μ_1, μ_2, and μ_f be the expected returns on the two risky assets and the return on the risk-free asset. Let σ_1 and σ_2 be the standard deviations of the returns on the two risky assets and let ρ_{12} be the correlation between the returns on the risky assets.

Define $V_1 = \mu_1 - \mu_f$ and $V_2 = \mu_2 - \mu_f$; V_1 and V_2 are called the "excess returns." Then the tangency portfolio uses weight

$$w_T = \frac{V_1\sigma_2^2 - V_2\rho_{12}\,\sigma_1\sigma_2}{V_1\sigma_2^2 + V_2\sigma_1^2 - (V_1+V_2)\rho_{12}\,\sigma_1\sigma_2}. \tag{5.1}$$

This formula will be derived in Section 5.6.6.

The tangency portfolio allocates a fraction w_T of the investment to the first risky asset and $(1 - w_T)$ to the second risky asset.

Let R_T, $E(R_T)$, and σ_T be the return, expected return, and standard deviation of the return on the tangency portfolio.

Example: Suppose as before that $\mu_1 = .14$, $\mu_2 = .08$, $\sigma_1 = .2$, $\sigma_2 = .15$, and $\rho_{12} = 0$. Suppose as well that $\mu_f = .06$. Then $V_1 = .14 - .06 = .08$ and $V_2 = .08 - .06 = .02$. Plugging these values into formula (5.1) we get $w_T = .693$. Therefore,

$$E(R_T) = (.693)(.14) + (.307)(.08) = .122,$$

and

$$\sigma_T = \sqrt{(.693)^2(.2)^2 + (.307)^2(.15)^2} = .146.$$

Let R be the return on the portfolio that allocates a fraction ω of the investment to the tangency portfolio and $1 - \omega$ to the risk-free asset.

Then $R = \omega R_T + (1-\omega)\mu_f = \mu_f + \omega(R_t - R_f)$ so that

$$E(R) = \mu_f + \omega\{E(R_T) - \mu_f\} \quad \text{and} \quad \sigma_R = \omega\sigma_T.$$

Continuation of previous example: What is the optimal investment with $\sigma_R = .05$?

Answer: The maximum expected return with $\sigma_R = .05$ mixes the tangency portfolio and the risk-free asset such that $\sigma_R = .05$. Since $\sigma_T = .146$, we have that $.05 = \sigma_R = \omega\,\sigma_T = .146\,\omega$, so that $\omega = .05/.146 = .343$ and $1 - \omega = .657$.

So 65.7% of the portfolio should be in the risk-free asset. 34.3% should be in the tangency portfolio. Thus (.343)(69.3%) = 23.7% should be in the first risky asset and (.343)(30.7%) = 10.5 should be in the second risky asset.

In summary

Asset	Allocation
risk-free	65.7%
risky 1	23.7%
risky 2	10.5%
Total	99.9%

The total is not quite 100% because of rounding errors.

Now suppose that you want a 10% expected return. Compare

- The best portfolio of only risky assets
- the best portfolio of the risky assets and the risk-free asset

Answer:

- (best portfolio of risky assets)
  - $.1 = w(.14) + (1-w)(.08)$ implies that $w = 1/3$.
  - This is the *only* portfolio of risky assets with $E(R) = .1$, so by default it is best.
  - Then

    $$\sigma_R = \sqrt{w^2(.2)^2 + (1-w)^2(.15)^2} = \sqrt{(1/9)(.2)^2 + 4/9(.15)^2} = .120.$$

- (best portfolio of the two risky assets and the risk-free asset)
  - $.1 = E(R) = .06 + .062\omega = .06 + .425\sigma_R$, since $\sigma_R = \omega\sigma_T$ or $\omega = \sigma_R/\sigma_T = \sigma_R/.146$.
  - This implies that $\sigma_R = .04/.425 = .094$ and $\omega = .04/.062 = .645$.

So combining the risk-free asset with the two risky assets reduces σ_R from .120 to .094 while maintaining $E(R)$ at .1. The reduction in risk is (.120 − .094)/.094 = 28%. Not a bad reduction in risk!

More on the example: What is the best we can do combining the risk-free asset with only one risky asset? Assume that we still want to have $E(R) = .1$

- Second risky asset with the risk-free

  - Since $\mu_f = .06 < .1$ and $\mu_2 = .08 < .1$, no portfolio with only the second risky asset and the risk-free asset will have an expected return of .1. Another way to appreciate this fact is to solve $.1 = \omega(.08) + (1-\omega)(.06) = .06 + .02\omega$ to get $\omega = 2$ and $1 - \omega = -1$. However, ω and $1 - \omega$ must both be between 0 and 1 unless one is permitted to sell short, which means selling an asset one does not yet owe and purchasing it later at delivery time.

- First risky asset with the risk-free
  - $.1 = \omega(.14) + (1-\omega)(.06) = .06 + \omega(.08)$ imples that $\omega = .04/.08 = 1/2$.
  - Then $\sigma_R = \omega(.20) = .10$ which is greater than .094, the smallest risk with two risky assets and the risk-free asset such that $E(R) = .1$.

The minimum value of σ_R under various combinations of available assets are given in Table 5.4.2.

Available Assets	**Minimum** σ_R
1st risky, risk-free	0.1
2nd risky, risk-free	–
Both riskies	0.12
All three	0.094

Table 5.1: Minimum value of σ_R as a function of the available assets.

5.4.3 Effect of ρ_{12}

Positive correlation between the two risky asets is bad. With positive correlation, then two assets tend to move together which increases the volatility of the portfolio. Conversely, negative correlation is good. If the assets are negatively correlated, a negative return of one tends to occur with a positive return of the other so the volatility of the portfolio decreases. Figure 5.4 shows the efficient frontier and tangency portfolio when $\mu_1 = .14$, $\mu_2 = .09$, $\sigma_1 = .2$, $\sigma_2 = .15$, and $\mu_f = .03$. The value of ρ_{12} is varied from .7 to $-.7$. Notice that the Sharpe's ratio of the tangency portfolio returns increases as ρ_{12} decreases. This means that when ρ_{12} is small then efficient portfolios have less risk for a given expected return compared to when ρ_{12} is large.

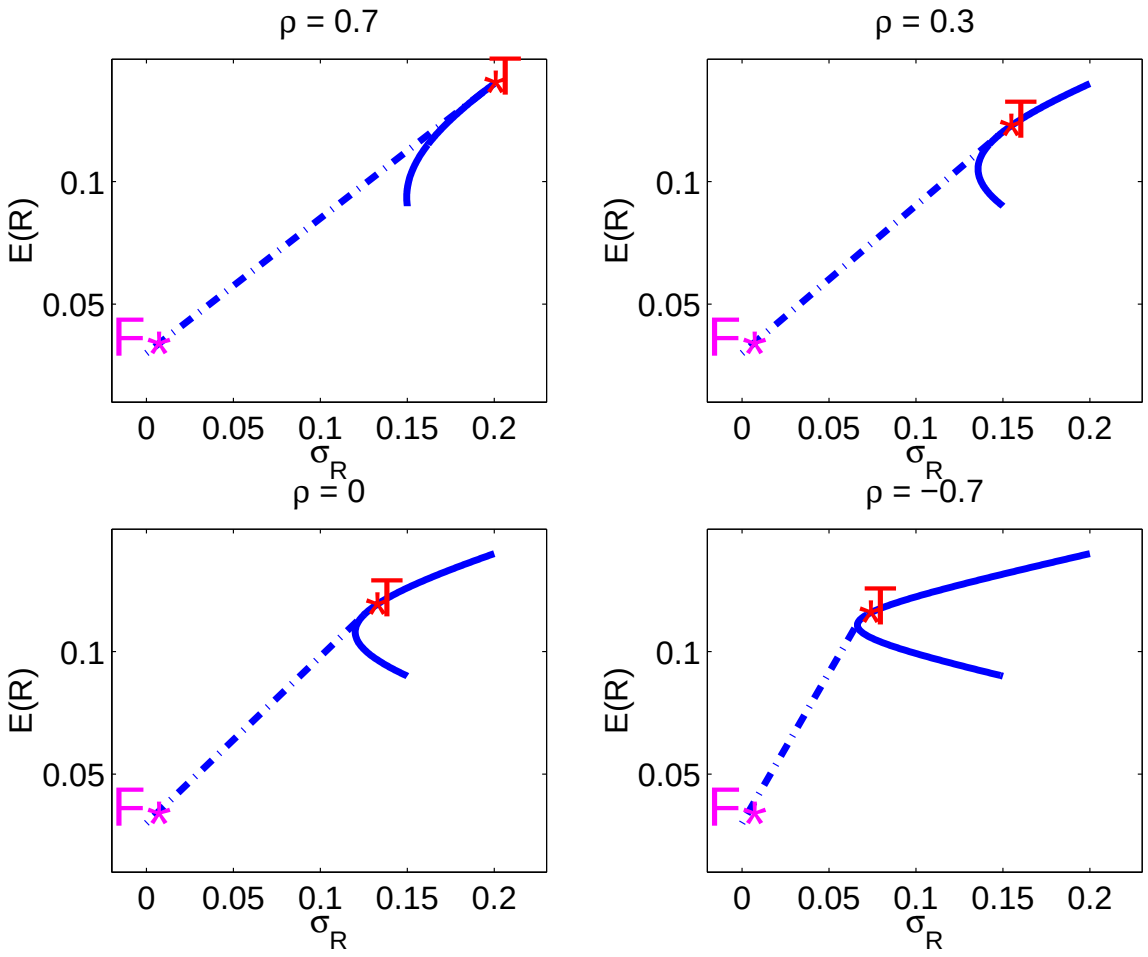


Figure 5.4: *Efficient frontier and tangency portfolio when* $\mu_1 = .14$, $\mu_2 = .09$, $\sigma_1 = .2$, $\sigma_2 = .15$, *and* $\mu_f = .03$. *The value of* ρ_{12} *is varied from .7 to* $-.7$.

5.5 Harry Markowitz

Chapter Two of *Capital Ideas: The Improbable Origins of Modern Wall Street* by Peter Bernstein is titled "Fourteen Pages to Fame." The title refers to the paper "Portfolio Selection" by Harry Markowitz that was published in the *Journal of Finance* in 1952. This article is indeed only fourteen pages though it was later expanded to the book *Portfolio Selection: Efficient Diversification of Investments* that was published by Markowitz in 1959.

Markowitz was not primarily interested in the stock market or investing. Rather, he was drawn to the more general issue of how people make trade-offs. Investors are faced with a trade-off between risk and expected return. The maxim "nothing ventured, nothing gained" isn't quite true, but risk-free rates of return can be smaller than many investors find acceptable. Markowitz's solution to the problem of risk also can be expressed as a maxim, "don't put all your eggs in one basket." (Keynes, would have agreed with Mark Twain who said, "put all your eggs in one basket — and then, **watch that basket!**"

Markowitz was born in 1927 and grow up in Chicago. His high school grades were not impressive, but he was intellectually curious and read a great deal on his own. At fourteen, he read Darwin's *Origin of Species* and later his hero was the 18th century Scottish philosopher David Hume. The knowledge he acquired on his own got him into the University of Chicago and even exempted him from the required science courses there. This self-study may have been ideal preparation for the highly original work that came later.

After graduation, Markowitz became a research associate at the Cowles Commission and a graduate student at his Alma Mater. While waiting outside his advisor's office one day, he began a conversation with a stock broker who suggested that he write his thesis on the stock market. Markowitz was somewhat surprised when later his advisor was enthusiastic over this idea.

Markowitz started to read what he could about investing. In the 1937 book *The Theory of Investment Value* by John Burr Williams, he found Williams's prescription for selecting stocks: one estimated the "intrinsic value" of a stock by forecasting all future dividends and calculating the "present value" all future dividends, that is, the discounted sum of all future dividends. William then recommends that one put all one's capital in the stock with the highest intrinsic value.

Markowitz had enough knowledge of the world to realize that this is not how investors actually operated. He had the key insight that humans

are risk-averse, and he began to explore the relationship between diversification and risk.

Interestingly, Markowitz did not recommend that expected returns be estimated from past data but rather from Williams's Dividend Discounted Model.

5.6 Risk-efficient portfolios with N risky assets

5.6.1 Efficient-portfolio mathematics

Efficient-portfolio mathematics generalizes our previous analysis with two risky assets to the more realistic case of many risky assets. This material is adopted from Section 5.2 of Campbell, Lo, and MacKinlay.

Assume that we have N risky assets and that the return on the ith risky asset is μ_i. Define

$$\boldsymbol{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix}$$

to be the random vector of returns. Then

$$E(\boldsymbol{R}) = \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}.$$

Let Ω_{ij} be the covariance between R_i and R_j. Also, let $\sigma_i = \sqrt{\Omega_{ii}}$ be the the standard deviation of R_i. Define $\rho_{ij} = \Omega_{ij}/(\sigma_i \sigma_j)$ as the correlation between R_i and R_j. Finally, let $\boldsymbol{\Omega}$ be the covariance matrix of $\boldsymbol{R}$, i.e.,

$$\boldsymbol{\Omega} = \text{COV}(\boldsymbol{R}),$$

so that the i, jth element of $\boldsymbol{\Omega}$ is $\Omega_{ij} = \text{Cov}(R_i, R_j)$.

Let

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}$$

be a matrix of portfolio weights and let

$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

be a column of N ones. We assume that[2] $\omega_1 + \cdots + \omega_N = \mathbf{1}^\mathsf{T}\boldsymbol{\omega} = 1$. The expected return on a portfolio with weights $\boldsymbol{\omega}$ is $\sum_{i=1}^N \omega_i\mu_i = \boldsymbol{\omega}^\mathsf{T}\boldsymbol{\mu}$.

When $N = 2$, $\omega_2 = 1 - \omega_1$. Suppose there is a target value, μ_P, of the expected return on the portfolio. We assume that

$$\min_{i=1,\ldots,N} \mu_i \leq \mu_P \leq \max_{i=1,\ldots,N} \mu_i,$$

since no portfolio can have an expected return higher than the individual asset with the highest expected return or smaller than the individual asset with the lowest expected return.[3] When $N = 2$ the target, μ_P, is achieved by only one portfolio and its ω_1 value solves

$$\mu_P = \omega_1\mu_1 + \omega_2\mu_2 = \mu_2 + \omega_1(\mu_1 - \mu_2).$$

For $N \geq 3$, there will be an infinite number of portfolios achieving the target, μ_P. The one with the smallest variance is called the "efficient" portfolio. Our goal is to find the efficient portfolio.

By equation (2.13), the variance of the return on the portfolio with weights $\boldsymbol{\omega}$ is

$$\sum_{i=1}^N \sum_{j=1}^N \omega_i\,\omega_j\,\Omega_{ij} = \boldsymbol{\omega}^\mathsf{T}\boldsymbol{\Omega}\boldsymbol{\omega}. \tag{5.2}$$

Thus, given a target μ_P, the efficent portfolio minimizes (5.2) subject to

$$\boldsymbol{\omega}^\mathsf{T}\boldsymbol{\mu} = \mu_P \tag{5.3}$$

and

$$\boldsymbol{\omega}^\mathsf{T}\mathbf{1} = 1. \tag{5.4}$$

We will denote the weights of the efficient portfolio by $\boldsymbol{\omega}_{\mu_P}$. To find $\boldsymbol{\omega}_{\mu_P}$, form the Lagrangian

$$L = \boldsymbol{\omega}^\mathsf{T}\boldsymbol{\Omega}\boldsymbol{\omega} + \delta_1(\mu_P - \boldsymbol{\omega}_{\mu_P}^\mathsf{T}\boldsymbol{\mu}) + \delta_2(1 - \boldsymbol{\omega}^\mathsf{T}\mathbf{1}).$$

Then solve

$$0 = \frac{\partial}{\partial\boldsymbol{\omega}}L = 2\boldsymbol{\Omega}\boldsymbol{\omega}_{\mu_P} - \delta_1\boldsymbol{\mu} - \delta_2\mathbf{1}. \tag{5.5}$$

[2]Recall that for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, their inner product is $\boldsymbol{a}^\mathsf{T}\boldsymbol{b} = \sum a_ib_i$. Therefore, $\boldsymbol{a}^\mathsf{T}\mathbf{1} = \sum a_i$.

[3]This is not true if one can buy risky assets on margin.

Definition: Here

$$\frac{\partial}{\partial \boldsymbol{\omega}} L = \begin{pmatrix} \partial L/\partial \omega_1 \\ \vdots \\ \partial L/\partial \omega_N \end{pmatrix}$$

means the gradient of L with respect to $\boldsymbol{\omega}$ with the other variables in L held fixed.

Fact: For an $n \times n$ matrix $\boldsymbol{A}$ and an n-dimensional vector $\boldsymbol{x}$,

$$\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{x}^\mathsf{T} \boldsymbol{A} \boldsymbol{x} = (\boldsymbol{A} + \boldsymbol{A}^\mathsf{T}) \boldsymbol{x}$$

The solution to (5.5) is

$$\boldsymbol{\omega}_{\mu_P} = \frac{1}{2} \boldsymbol{\Omega}^{-1} (\delta_1 \boldsymbol{\mu} + \delta_2 \mathbf{1}) = \boldsymbol{\Omega}^{-1} (\lambda_1 \boldsymbol{\mu} + \lambda_2 \mathbf{1})$$

where λ_1 and λ_2 are new Lagrange multipliers:

$$\lambda_1 = \frac{1}{2} \delta_1 \text{ and } \lambda_2 = \frac{1}{2} \delta_2.$$

Thus,

$$\boldsymbol{\omega}_{\mu_P} = \lambda_1 \boldsymbol{\Omega}^{-1} \boldsymbol{\mu} + \lambda_2 \boldsymbol{\Omega}^{-1} \mathbf{1},$$

where λ_1 and λ_2 are yet to be determined scaler quantities. We need to use the constraints to determine λ_1 and λ_2. Therefore,

$$\mu_p = \boldsymbol{\mu}^\mathsf{T} \boldsymbol{\omega}_{\mu_P} = \lambda_1 \boldsymbol{\mu}^\mathsf{T} \boldsymbol{\Omega}^{-1} \boldsymbol{\mu} + \lambda_2 \boldsymbol{\mu}^\mathsf{T} \boldsymbol{\Omega}^{-1} \mathbf{1}, \tag{5.6}$$

and

$$1 = \mathbf{1}^\mathsf{T} \boldsymbol{\omega}_{\mu_P} = \lambda_1 \mathbf{1}^\mathsf{T} \boldsymbol{\Omega}^{-1} \boldsymbol{\mu} + \lambda_2 \mathbf{1}^\mathsf{T} \boldsymbol{\Omega}^{-1} \mathbf{1}. \tag{5.7}$$

These are equations in λ_1 and λ_2. We will introduce simpler notation for the coefficients:

$$\begin{aligned} A &= \boldsymbol{\mu}^\mathsf{T} \boldsymbol{\Omega}^{-1} \mathbf{1} = \mathbf{1}^\mathsf{T} \boldsymbol{\Omega}^{-1} \boldsymbol{\mu} && (5.8) \\ B &= \boldsymbol{\mu}^\mathsf{T} \boldsymbol{\Omega}^{-1} \boldsymbol{\mu}, && (5.9) \\ C &= \mathbf{1}^\mathsf{T} \boldsymbol{\Omega}^{-1} \mathbf{1}. && (5.10) \end{aligned}$$

Then (5.6) and (5.7) can be rewritten as

$$\begin{aligned} \mu_P &= B\lambda_1 + A\lambda_2 \\ 1 &= A\lambda_1 + C\lambda_2. \end{aligned}$$

Let

$$D = BC - A^2 \tag{5.11}$$

be the determinant of this system of linear equations. The solution is

$$\lambda_1 = \frac{-A + C\,\mu_p}{D} \text{ and } \lambda_2 = \frac{B - A\,\mu_P}{D}.$$

It follows after some algebra that

$$\boldsymbol{\omega}_{\mu_P} = \boldsymbol{g} + \boldsymbol{h}\,\mu_P, \tag{5.12}$$

where

$$\boldsymbol{g} = \frac{B\,\boldsymbol{\Omega}^{-1}\mathbf{1} - A\,\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}{D} = \frac{B}{D}\boldsymbol{\Omega}^{-1}\mathbf{1} - \frac{A}{D}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}, \tag{5.13}$$

and

$$\boldsymbol{h} = \frac{C\,\boldsymbol{\Omega}^{-1}\boldsymbol{\mu} - A\,\boldsymbol{\Omega}^{-1}\mathbf{1}}{D} = \frac{C}{D}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu} - \frac{A}{D}\boldsymbol{\Omega}^{-1}\mathbf{1}. \tag{5.14}$$

Notice that $\boldsymbol{g}$ and $\boldsymbol{h}$ are fixed vectors, since they depend on the fixed vector $\boldsymbol{\mu}$ and the fixed matrix $\boldsymbol{\Omega}$. Also, the scalars A, C, and D are functions of $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$ so they are also fixed. The target expected return, μ_P, can be varied over some range of values, for example,

$$\min_{i=1,\ldots,N} \mu_i \leq \mu_P \leq \max_{i=1,\ldots,N} \mu_i$$

or

$$\mu_{\min} \leq \mu_P \leq \max_{i=1,\ldots,N} \mu_i \tag{5.15}$$

where $\mu_{\min}$ the expected return of minimum variance portfolio that is given in equation (5.18) in the next section.

As μ_P varies over the range (5.15), we get a locus $\boldsymbol{\omega}_{\mu_P}$ of efficient portfolios called the "efficient frontier." We can illustrate the efficient frontier by the following algorithm:

1. Vary μ_P along a grid. For each value of μ_P on this grid, compute σ_{μ_P} by:
   (a) computing $\boldsymbol{\omega}_{\mu_P} = \boldsymbol{g} + \boldsymbol{h}\,\mu_P$
   (b) then computing $\sigma_{\mu_P} = \sqrt{\boldsymbol{\omega}_{\mu_P}^{\mathsf{T}}\boldsymbol{\Omega}\boldsymbol{\omega}_{\mu_P}}$
2. Plot the values (μ_P, σ_{μ_P}). The values (μ_P, σ_{μ_P}) with $\mu_P \geq \mu_{\min}$ are the efficient frontier. The other values of (μ_P, σ_{μ_P}) lie below the efficient frontier and are (very) inefficient portfolios.

This algorithm is implemented in the MATLAB program "portfolio03.m" on the course's web site and listed below:

```
% portfolio03.m
% Matlab program to plot efficient frontier and minimum
% variance portfolio

% Any text the follows a percent sign (like this text) is a comment
% Input mean vector and covariance matrix of returns here

% I am trying to follow the notation in the lecture notes.
% A "b" in front of a name means "bold face"

bmu = [.08;.03;.05] ;
bOmega = [ .3 .02 .01 ;
  .02 .15 .03 ;
  .01 .03 .18 ] ;

bone = ones(length(bmu),1) ; % Define vector of ones

ibOmega = inv(bOmega) ; % Invert Omega - "i" means inverse
A = bone'*ibOmega*bmu ;
B = bmu'*ibOmega*bmu ;
C = bone'*ibOmega*bone ;
D = B*C - A^2 ;
bg = (B*ibOmega*bone - A*ibOmega*bmu)/D ;
bh = (C*ibOmega*bmu - A*ibOmega*bone)/D ;

% Compute minimum expected return and minimum return SD
gg = bg'*bOmega*bg ;
hh = bh'*bOmega*bh ;
gh = bg'*bOmega*bh ;
mumin = - gh/hh ;
sdmin = sqrt( gg * ( 1 - gh^2/(gg*hh)) ) ;

muP = linspace(min(bmu),max(bmu),50) ; % muP grid
sigmaP = zeros(1,50) ; % Storage


for i=1:50 ;
omegaP = bg + muP(i)*bh ;
sigmaP(i) = sqrt(omegaP'*bOmega*omegaP) ;
end ;

fsize = 16 ;
ind = (muP > mumin) ; % Indicates efficient horizon
ind2 = (muP < mumin) ; % Indicates locus below efficient horizon

% Create plot - efficient horizon is shown as a solid curve
```

```
% -  the inefficient part of the locus is dashed

p1 = plot(sigmaP(ind),muP(ind),'-',sigmaP(ind2),muP(ind2),'--' , ...
sdmin,mumin,'.') ;

% Change line widths, marker sizes, and colors for better appearance
set(p1(1:2),'linewidth',4) ;
set(p1(1:2),'color','blue') ;
set(p1(3),'markersize',40) ;
set(p1(3),'color','red') ;

% Label axes
xlabel('standard deviation of return','fontsize',fsize) ;
ylabel('expected return','fontsize',fsize) ;
set(gca,'xlim',[0, .5]) ;
set(gca,'ylim',[0, .08]) ;
grid ;

% Print plot as a color postscript file
print portfolio03.ps -depsc ;
!mv portfolio03.ps ~/public_html/or473/LectNotes/portfolio03.ps ;
```

To use this program replace `bmu` and `bOmega` in the program by the vector of expected returns and covariance matrix of returns for the assets you wish to analyze. Figure 5.5 is the output from this program. Figure 5.6 is an enhancement of Figure 5.5 produced by program "portfolio02.m." Figure 5.6 shows the three efficient frontiers for only a pair of assets.

5.6.2 The minimum variance portfolio

We just showed that the efficient portfolio with expected return equal to μ_P has weights

$$\boldsymbol{\omega}_{\mu_P} = \boldsymbol{g} + \boldsymbol{h}\,\mu_P, \tag{5.16}$$

The variance of this portfolio is

$$\begin{aligned} \text{Var}(\mu_P) &= (\boldsymbol{g} + \boldsymbol{h}\,\mu_P)^{\mathsf{T}} \boldsymbol{\Omega} (\boldsymbol{g} + \boldsymbol{h}\,\mu_P) \\ &= \boldsymbol{g}^{\mathsf{T}} \boldsymbol{\Omega} \boldsymbol{g} + 2\boldsymbol{g}^{\mathsf{T}} \boldsymbol{\Omega} \boldsymbol{h} \mu_P + \boldsymbol{h}^{\mathsf{T}} \boldsymbol{\Omega} \boldsymbol{h} \mu_P^2. \end{aligned} \tag{5.17}$$

To find the minimum variance portfolio we minimize this quantity over μ_P by solving

$$0 = \frac{d}{d\mu_p} \text{Var}(\mu_P) = 2\boldsymbol{g}^{\mathsf{T}} \boldsymbol{\Omega} \boldsymbol{h} + 2\boldsymbol{h}^{\mathsf{T}} \boldsymbol{\Omega} \boldsymbol{h} \mu_P.$$

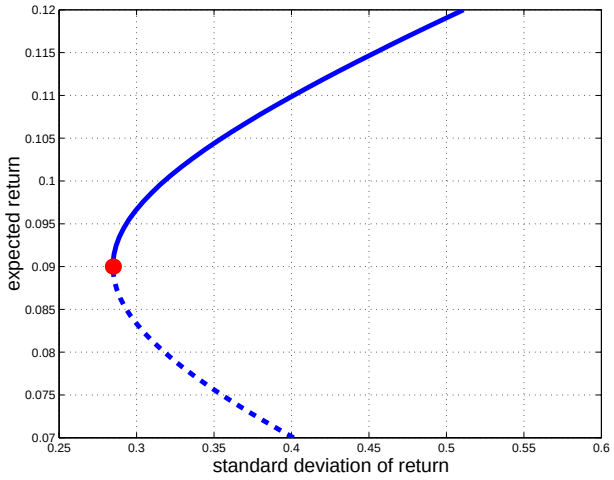


Figure 5.5: *Efficient frontier (solid) plotted for $N = 3$ assets by the program "portfolio03.m."*

The solution is the expected return of the minimum variance portfolio given by

$$\mu_{\min} := -\frac{\boldsymbol{g}^{\mathsf{T}}\boldsymbol{\Omega}\boldsymbol{h}}{\boldsymbol{h}^{\mathsf{T}}\boldsymbol{\Omega}\boldsymbol{h}}. \tag{5.18}$$

Plugging $\mu_{\min}$ into (5.17) we find that the smallest possible variance of a portfolio is

$$\mathrm{Var}(\mu_{\min}) = \boldsymbol{g}^{\mathsf{T}}\boldsymbol{\Omega}\boldsymbol{g} - \frac{(\boldsymbol{g}^{\mathsf{T}}\boldsymbol{\Omega}\boldsymbol{h})^2}{\boldsymbol{h}^{\mathsf{T}}\boldsymbol{\Omega}\boldsymbol{h}}.$$

The portfolio weights as functions of μ_P are plotted in Figure 5.7. The weights can be negative. Negative weights can be obtained by the technique of selling short which is described in Section 5.6.3.

If one wants to avoid short selling, then one must impose the additional constraints that $w_i \geq 0$ for $i = 1, \ldots, N$. Minimization of portfolio risk subject to $\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\mu} = \mu_P$, $\boldsymbol{\omega}^{\mathsf{T}}\mathbf{1} = 1$, and these additional nonnegativity constraints is a quadratic programming problem. (This minimization problem cannot be solved by the method of Lagrange multipliers because of the inequality constraints.) Quadratic programming algorithms are not hard to find. For example, the program "quadprog" in MATLAB's Optimization Toolbox does quadratic programming.

Figure 5.8 and 5.9 were produced by the program "portfolio02QP.m" listed starting on page 140 that uses "quadprog" in MATLAB. Quadratic

programming in MATLAB and "portfolio02QP.m are discussed in Section 5.9. Figure 5.8 shows two efficient frontiers. The dashed curve is the nonconstrainted efficient frontier and is the same the effient frontier in Figure 5.5. The second is the efficient frontier with weights constrained to be nonnegative using quadratic programming. The portfolio weights with the nonnegativity constraint are plotted as functions of μ_P in Figure 5.9.

Now suppose that we have a risk-free asset and we want to mix the risk-free asset with *some* efficient portfolio. One can see geometrically that there is a tangency portfolio; see Figure 5.10. **The optimal portfolio *always* is a mixture of the risk-free asset with the tangency portfolio.** This is a remarkable simplification.

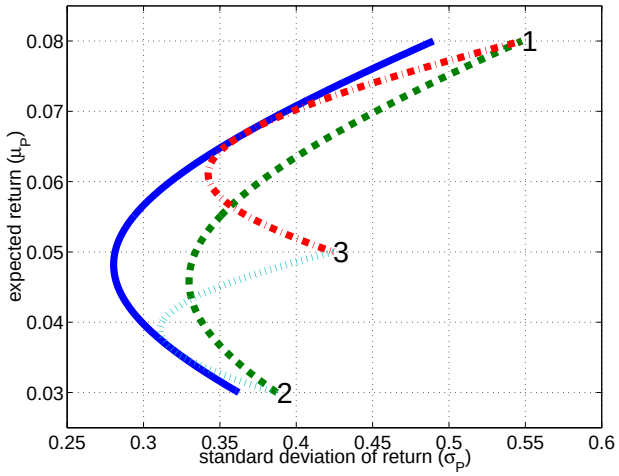


Figure 5.6: *Efficient frontier (solid) plotted for $N = 3$ assets by the program "portfolio02.m." "1," "2," and "3" are the three single assets. The efficient frontiers for just two assets are dashed (1 and 2), dashed-and-dotted (1 and 3), and dotted (2 and 3).*

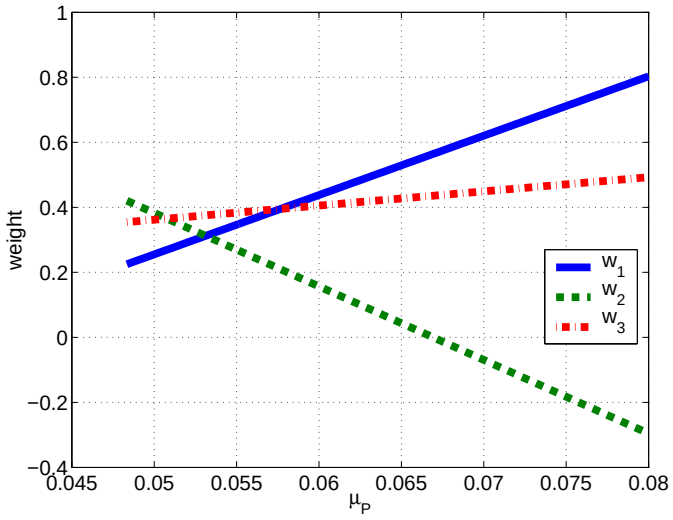


Figure 5.7: *Weights for assets 1, 2, and 3 as functions of $\mu_p \geq \mu_{\min}$. Note that the weights for assets 1 and 2 can be negative, so that short selling would be required. Produced by program "portfolio02.m."*

5.6.3 Selling short

Selling short is a way to profit if a stock price goes *down*. To sell a stock short, one sells the stock without owning it. The stock must be borrowed from a broker or another customer of the broker. At a later point in time, one buys the sale and gives it back to the lender. This closes the short position.

Suppose a stock is selling at \$25/share and you sell 100 shares short. This gives you \$2,500. If the goes down to \$17 share, you can buy the 100 shares for \$1,700 and close out your short position.

Suppose that you have \$100 and there are two risky assets. With your money you could buy \$150 worth of risky asset 1 and sell \$50 short of risky asset 2. The net cost would be exactly \$100. If R_1 and R_2 are the returns on risky assets 1 and 2, then the return on your portfolio would be

$$\frac{3}{2}R_1 + \left(-\frac{1}{2}\right) R_2.$$

Your portfolio weights are $w_1 = 3/2$ and $w_2 = -1/2$. Thus, you hope that risky asset 1 rises in price and risky asset 2 falls in price. Here, as elsewhere, we have ignored transaction costs.

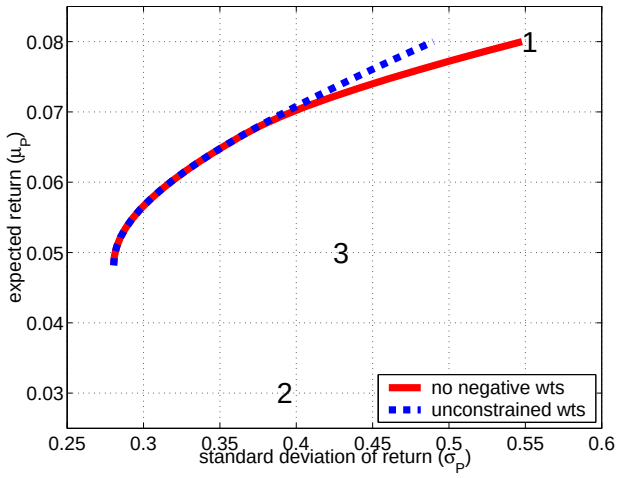


Figure 5.8: *Efficient frontier plotted by the program "portfolio02QP.m" for $N = 3$ assets. "1," "2," and "3" are the three single assets. The efficient frontiers are found with and without the constraint of no negative weights. The constrained efficient frontier is computed using MATLAB's quadratic programming algorithm.*

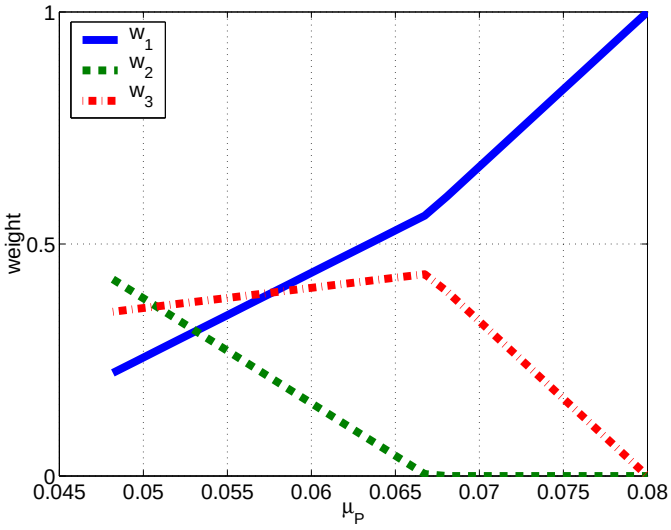


Figure 5.9: *Weights for assets 1, 2, and 3 as functions of $\mu_p \geq \mu_{\min}$. The weights for all three assets are constrained to be nonnegative.*

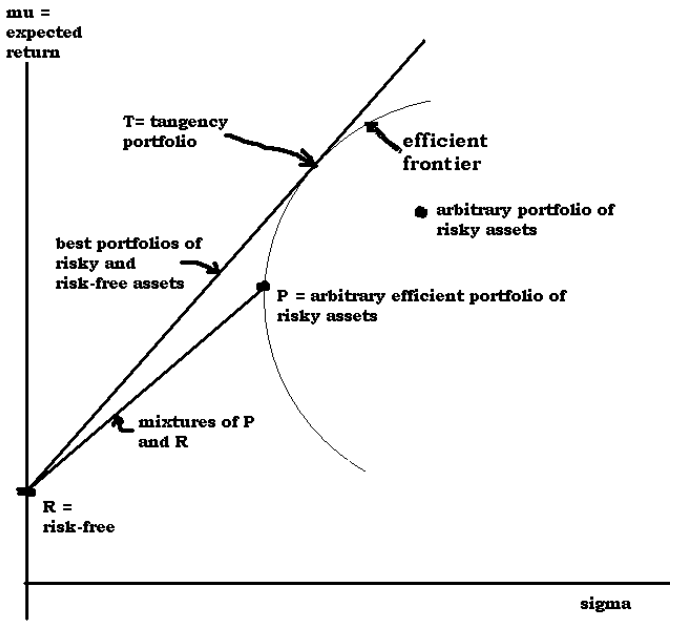


Figure 5.10: *Finding the best portfolios that combine risky and risk-free assets. R is the risk-free asset. T is the tangency portfolio. The optimal portfolios are on the line connecting R and T. The efficient frontier gives the set of optimal portfolios of risky assets.*

5.6.4 The Interior decorator fallacy

It is often thought that a stock portfolio should be tailored to the financial circumstances of a client, as an interior decorator furnishes your home to suit your tastes. For example, widows and orphans should hold conservative "income stocks," or so it is said.

Bernstein, in his book *Capital Ideas*, calls this the "interior decorator fallacy." Bernstein tells the story of a woman in her forties who came to him in 1961 for investment advice. She was married to a clergyman with a modest income. She had just inherited money which she wanted to invest. Bernstein recommended a portfolio that included stocks with good growth potential but low dividends, e.g., Georgia Pacific, IBM, and Gillette. The client was worried that these were too risky, but she eventually took Bernstein's advice, which turned out to be sound. Bernstein reasoned that even someone with modest means should benefit from the long-term growth potential of these "hot" stocks.

In another case, Bernstein recommended electric utilities, a conservative choice, to a young business excecutive who wanted a more aggressive portfolio. Again, this recommendation was at odds with conventional wisdom.

A new view, based both on mathematical theory and experience, is that there is a best portfolio (the tangency portfolio) that is the same for everyone. An individual's circumstances only determines the appropriate mix between risk-free assets and the tangency portfolio. The clergyman's wife should invest a higher percentage of her money in risk-free assets than the young business executive. In 1961, Bernstein had the right intuition but he had not yet heard of the efficient frontier or the tangency portfolio.

5.6.5 Back to the math—finding the tangency portfolio

Here's the mathematics behind Figure 5.10. We now remove the assumption that $\boldsymbol{\omega}^\mathsf{T}\mathbf{1} = 1$. The quantity $1 - \boldsymbol{\omega}^\mathsf{T}\mathbf{1}$ is invested in the risk-free asset. (Does it make sense to have $1 - \boldsymbol{\omega}^\mathsf{T}\mathbf{1} < 0$?). The expected return is

$$\boldsymbol{\omega}^\mathsf{T}\boldsymbol{\mu} + (1 - \boldsymbol{\omega}^\mathsf{T}\mathbf{1})\mu_f, \tag{5.19}$$

where μ_f is the return on the risk-free asset. The constraint to be satisfied is that (5.19) is equal to μ_P. Thus, the Lagrangian function is

$$L = \boldsymbol{\omega}^\mathsf{T}\boldsymbol{\Omega}\boldsymbol{\omega} + \delta\{\mu_P - \boldsymbol{\omega}^\mathsf{T}\boldsymbol{\mu} - (1 - \boldsymbol{\omega}^\mathsf{T}\mathbf{1})\mu_f\}.$$

Here δ is a Lagrange multiplier. Since

$$0 = \frac{\partial}{\partial \boldsymbol{\omega}} L = 2\boldsymbol{\Omega}\boldsymbol{\omega} + \delta(-\boldsymbol{\mu} + \mathbf{1}\mu_f),$$

the optimal weight vector, i.e., the vector of weights that minimizes risk subject to the constraint on the expected return, is

$$\boldsymbol{\omega}_{\mu_P} = \lambda \boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1}), \tag{5.20}$$

where $\lambda = \delta/2$. To find λ, we use our constraint:

$$\boldsymbol{\omega}_{\mu_P}^{\mathsf{T}} \boldsymbol{\mu} + (1 - \boldsymbol{\omega}_{\mu_P}^{\mathsf{T}} \mathbf{1})\mu_f = \mu_P. \tag{5.21}$$

Rearranging (5.21), we get

$$\boldsymbol{\omega}_{\mu_P}^{\mathsf{T}} (\boldsymbol{\mu} - \mu_f \mathbf{1}) = \mu_P - \mu_f. \tag{5.22}$$

Therefore, substituting (5.20) into (5.22) we have

$$\lambda(\boldsymbol{\mu} - \mu_f \mathbf{1})^{\mathsf{T}} \boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1}) = \mu_P - \mu_f,$$

or

$$\lambda = \frac{\mu_P - \mu_f}{(\boldsymbol{\mu} - \mu_f)^{\mathsf{T}} \boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1})}. \tag{5.23}$$

Then substituting (5.23) into (5.20)

$$\boldsymbol{\omega}_{\mu_P} = c_P \, \overline{\boldsymbol{\omega}},$$

where

$$c_P = \frac{\mu_P - \mu_f}{(\boldsymbol{\mu} - \mu_f \mathbf{1})^{\mathsf{T}} \boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1})}$$

and

$$\overline{\boldsymbol{\omega}} = \boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1}). \tag{5.24}$$

Note that $(\boldsymbol{\mu} - \mu_f \mathbf{1})$ is the vector of "excess returns," that is, the amount by which the expected returns on the risky assets exceed the risk-free return. The excess returns measure how much the market pays for assuming risk.

$\overline{\boldsymbol{\omega}}$ is not quite a portfolio because these weights do not necessarily sum to one. The tangency portfolio is a scalar multiple of $\overline{\boldsymbol{\omega}}$:

$$\boldsymbol{\omega}_T = \frac{\overline{\boldsymbol{\omega}}}{\mathbf{1}^{\mathsf{T}} \overline{\boldsymbol{\omega}}}. \tag{5.25}$$

The optimal weight vector $\boldsymbol{\omega}_{\mu_P}$ can be expressed in terms of the tangency portfolio as $\boldsymbol{\omega}_{\mu_P} = c_p\,\overline{\boldsymbol{\omega}} = c_p(\mathbf{1}^{\mathsf{T}}\overline{\boldsymbol{\omega}})\boldsymbol{\omega}_T$. Therefore, $c_P\,(\mathbf{1}^{\mathsf{T}}\overline{\boldsymbol{\omega}})$ tells us how much weight to put on the tangency portfolio, $\boldsymbol{\omega}_T$. The amount of weight to put of the risk-free asset is $= 1 - c_p\,(\overline{\boldsymbol{\omega}}^{\mathsf{T}}\mathbf{1})$.

Note that $\overline{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}_T$ do not depend on μ_p.

The MATLAB program "portfolio04.m" on the course web site is an extension of "portfolio03.m." portfolio03.m, which is listed below, also plots of the tangency portfolio (T) and the line connecting the risk-free asset (F) with the tangency portfolio.

```
% portfolio04 - extension of portfolio03
% Input mean vector and covariance matrix of returns here

bmu = [.08;.03;.05] ;
bOmega = [ .3 .02 .01 ;
  .02 .15 .03 ;
  .01 .03 .18 ] ;

bOmega = diag(diag(bOmega)) ;

muf = .02 ;

bone = ones(length(bmu),1) ;


muP = linspace(min(bmu),max(bmu),50) ;
sigmaP = zeros(1,50) ;

ibOmega = inv(bOmega) ;
A = bone'*ibOmega*bmu ;
B = bmu'*ibOmega*bmu ;
C = bone'*ibOmega*bone ;
D = B*C - A^2 ;
bg = (B*ibOmega*bone - A*ibOmega*bmu)/D ;
bh = (C*ibOmega*bmu - A*ibOmega*bone)/D ;

for i=1:50 ;
omegaP = bg + muP(i)*bh ;
sigmaP(i) = sqrt(omegaP'*bOmega*omegaP) ;
end ;


% Compute minimum expected return and minimum return SD
gg = bg'*bOmega*bg ;
hh = bh'*bOmega*bh ;
gh = bg'*bOmega*bh ;
mumin = - gh/hh ;
```

```
sdmin = sqrt( gg * ( 1 - gh^2/(gg*hh)) ) ;


bomegabar = ibOmega*(bmu - muf*bone) ;
bomegaT = bomegabar/(bone'*bomegabar) ;
sigmaT = sqrt(bomegaT'*bOmega*bomegaT) ;
muT = bmu'*bomegaT ;

fsize = 16 ;
fsize2 = 35 ;
bomegaP2 = [0;.3;.7] ;
sigmaP2 = sqrt(bomegaP2'*bOmega*bomegaP2) ;
muP2 = bmu'*bomegaP2 ;
ind = (muP > mumin) ; % Indicates efficient horizon
ind2 = (muP < mumin) ; % Indicates locus below efficient horizon

clf ;
p1 = plot(sigmaP(ind),muP(ind),'-',sigmaP(ind2),muP(ind2), ...
'--' ,sdmin,mumin,'.') ;
l1 = line([0,sigmaT],[muf,muT]) ;
t1= text(sigmaP2,muP2,'* P','fontsize',fsize2) ;
t2= text(sigmaT,muT,'* T','fontsize',fsize2) ;
t3=text(.01,muf+.006,'F','fontsize',fsize2) ;
t3B= text(0,muf,'*','fontsize',fsize2) ;

set(p1(1:2),'linewidth',4) ;
set(p1(3),'markersize',40) ;
set(p1(3),'color','red') ;
set(l1,'linewidth',4) ;
set(l1,'linestyle','--') ;

xlabel('standard deviation of return','fontsize',fsize) ;
ylabel('expected return','fontsize',fsize) ;
grid ;

print portfolio04.ps -depsc ;
!mv portfolio04.ps ~/public_html/or473/LectNotes/portfolio04.ps ;
```

5.6.6 Example: deriving formulas for $N = 2$

If $N = 2$, then

$$\mathbf{\Omega} = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

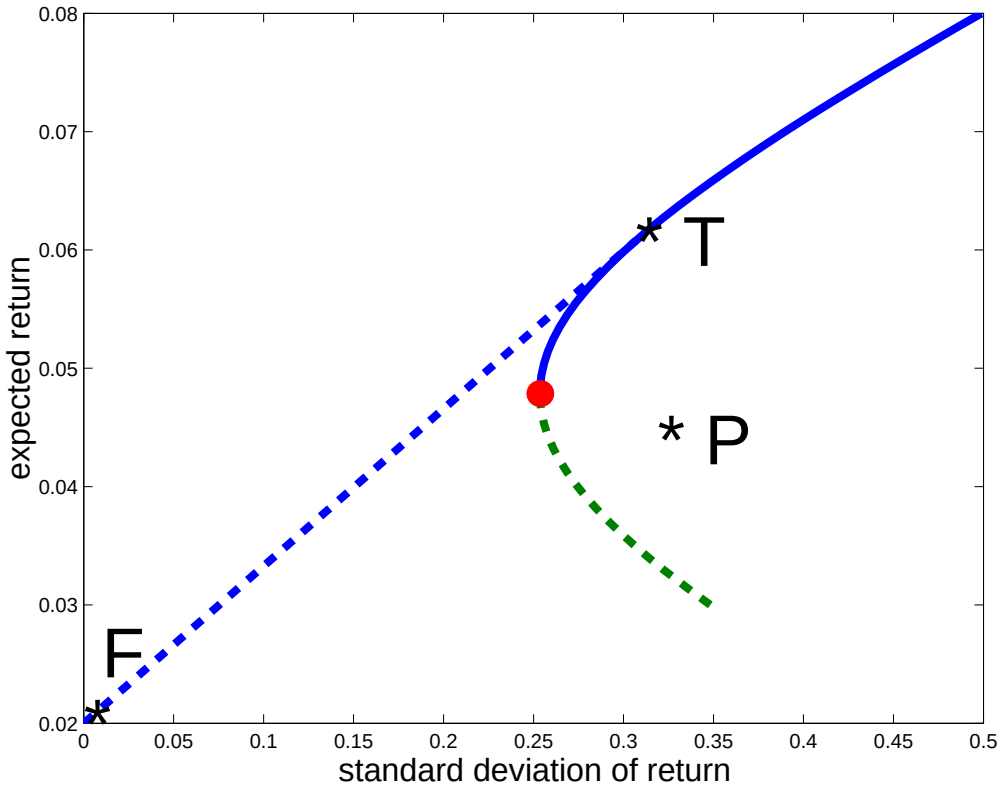


Figure 5.11: *Efficient frontier and line of optimal combinations of risky and risk-free assets plotted by the program "portfolio04.m" for $N = 3$ assets. "P" is the portfolio with weights (0 .3 .7) that is an example of a portfolio not on the effcient frontier. "T" is the tangency portfolio and "F" is the risk-free asset.*

You should check that

$$\mathbf{\Omega}^{-1} = \frac{1}{1-\rho_{12}^2} \begin{pmatrix} \sigma_1^{-2} & -\rho_{12}\sigma_1^{-1}\sigma_2^{-1} \\ -\rho_{12}\sigma_1^{-1}\sigma_2^{-1} & \sigma_2^{-2} \end{pmatrix}.$$

Also,

$$\boldsymbol{\mu} - \mu_f \mathbf{1} = \begin{pmatrix} \mu_1 - \mu_f \\ \mu_2 - \mu_f \end{pmatrix}.$$

Therefore,

$$\overline{\boldsymbol{\omega}} = \mathbf{\Omega}^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1}) = \frac{1}{1-\rho_{12}^2} \begin{pmatrix} \frac{\mu_1-\mu_f}{\sigma_1^2} - \frac{\rho_{12}(\mu_2-\mu_f)}{\sigma_1\sigma_2} \\ -\frac{\rho_{12}(\mu_1-\mu_f)}{\sigma_1\sigma_2} + \frac{\mu_2-\mu_f}{\sigma_2^2} \end{pmatrix}.$$

Next, let $V_1 = \mu_1 - \mu_f$ and $V_2 = \mu_2 - \mu_f$. Then,

$$\mathbf{1}^\mathsf{T}\overline{\boldsymbol{\omega}} = \frac{V_1\,\sigma_2^2 + V_2\,\sigma_1^2 - (V_1 + V_2)\,\rho_{12}\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2(1-\rho_{12})}.$$

It follows that

$$\frac{\overline{\boldsymbol{\omega}}}{\mathbf{1}^\mathsf{T}\overline{\boldsymbol{\omega}}} = \frac{1}{V_2\,\sigma_2 + V_2\,\sigma_1^2 - (V_1 + V_2)\,\rho_{12}\sigma_1\sigma_2}\begin{pmatrix} V_1\sigma_2^2 - V_2\,\rho_{12}\sigma_1\sigma_2 \\ V_2\,\sigma_1^2 - V_1\,\rho_{12}\sigma_1\sigma_2 \end{pmatrix}.$$

Compare the first element of this vector with (5.1), the formula that gives the weight of the first of two risky assets in the tangency portfolio.

5.6.7 A numerical example with $N = 3$

In this section, a numerical example is given of computing the efficient frontier, minimum variance porftolio, and tangency portfolio when N is 3.

In principle, one could find formulas analogous to (5.1) for cases where N is bigger than 2, but they would be cumbersome. It is better to use matrix based formulas such as (5.25).

In this section we work through a numerical example with $N = 3$. In practice, we would want to do these calculations using a computer program and, in fact, they really were done with a program. However, I will go through the calculations as if they were done "by hand" as a illustration of the mathematics.

Assume that the mean returns are .07, .12, and .09. Assume as well that the variances of the returns are .2, .3, and .25. Finally, to make the calculations simple, assume that the returns are uncorrelated. The mean vector is

$$\boldsymbol{\mu} = \begin{pmatrix} .07 \\ .12 \\ .09 \end{pmatrix}.$$

Because the returns are uncorrelated the covariance matrix is the diagonal matrix

$$\boldsymbol{\Omega} = \begin{pmatrix} .2 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & .25 \end{pmatrix}.$$

Diagonal matrices are easy to invert and the inverse covariance matrix is

$$\boldsymbol{\Omega}^{-1} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Then using equations (5.8)–(5.11)

$$A = \boldsymbol{\mu}^{\mathsf{T}}\boldsymbol{\Omega}^{-1}\mathbf{1} = \begin{pmatrix} .07 & .12 & .09 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1.1100,$$

$$B = \boldsymbol{\mu}^{\mathsf{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu} = \begin{pmatrix} .07 & .12 & .09 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} .07 \\ .12 \\ .09 \end{pmatrix} = .1049,$$

and

$$C = \mathbf{1}^{\mathsf{T}}\boldsymbol{\Omega}^{-1}\mathbf{1} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 12.3333,$$

Also $D = BC - A^2 = .0617$. Then by (5.13)

$$\begin{aligned} \boldsymbol{g} &= \frac{.1049}{.0617} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &- \frac{1.1100}{.0617} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} .07 \\ .12 \\ .09 \end{pmatrix} = \begin{pmatrix} 2.2054 \\ -1.5297 \\ .3243 \end{pmatrix}, \end{aligned}$$

and by equation (5.14)

$$\begin{aligned} \boldsymbol{h} &= \frac{12.3333}{.0617} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} .07 \\ .12 \\ .09 \end{pmatrix} \\ &- \frac{1.1100}{.0617} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -20 \\ 20 \\ 0 \end{pmatrix}. \end{aligned}$$

Then by equation (5.12) the efficient weight vector for a portfolio with expected return equal to μ_P is

$$\boldsymbol{\omega}_{\mu_P} = \boldsymbol{g} + \boldsymbol{h}\,\mu_P = \begin{pmatrix} 2.2054 - 20\mu_P \\ -1.5297 + 20\mu_P \\ .3243 \end{pmatrix}.$$

In other words, if we select a target expected return μ_P, these are the weights that give us that expected return with the smallest possible risk. As μ_P

changes, the first and second components of $\boldsymbol{\omega}_{\mu_P}$ change. The third component of $\boldsymbol{\omega}_{\mu_P}$ does not change in this example because this is a somewhat unusual case where one of the components of $\boldsymbol{h}$ is exactly zero. Notice what happens as we *increase* μ_P: the first component of $\boldsymbol{\omega}_{\mu_P}$ *decreases*, the second component *increases*, and the third component *stays the same.* This makes perfect sense. The first risky asset has the smallest expected return and the second risky asset has the greatest expected return. Thus, to increase the portfolio's expected return we move some of our allocation from the first risky asset to the second.

To find the minimum variance portfolio, note that

$$\boldsymbol{g}^\mathsf{T}\boldsymbol{\Omega}\boldsymbol{h} = -18$$

and

$$\boldsymbol{h}^\mathsf{T}\boldsymbol{\Omega}\boldsymbol{h} = 200.$$

Then by (5.18)

$$\mu_{\min} = -\frac{\boldsymbol{g}^\mathsf{T}\boldsymbol{\Omega}\boldsymbol{g}}{\boldsymbol{h}^\mathsf{T}\boldsymbol{\Omega}\boldsymbol{h}} = 0.0900.$$

The weights for the minimum variance portfolio are

$$\boldsymbol{g} + \boldsymbol{h}\,\mu_{\min} = \begin{pmatrix} .4054 \\ .2703 \\ .3243 \end{pmatrix}.$$

Now let's find the tangency portfolio when $\mu_f = .05$. We have by (5.24)

$$\overline{\boldsymbol{\omega}} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} .07 - .05 \\ .12 - .05 \\ .09 - .05 \end{pmatrix} = \begin{pmatrix} .1000 \\ .2333 \\ .1600 \end{pmatrix}$$

and by (5.25)

$$\boldsymbol{\omega}_T = \frac{1}{.1000 + .2333 + .1600} \begin{pmatrix} .1000 \\ .2333 \\ .1600 \end{pmatrix} = \begin{pmatrix} .2027 \\ .4730 \\ .3243 \end{pmatrix}.$$

5.7 Is the theory useful?

This theory of portfolio selection could be used if N were small. We would need estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$. These would be obtained from recent returns

data. Of course, there is no guarantee that future returns will behave like returns in the past, but this is the working assumption.

The next section gives an example of using portfolio theory to allocate capital among eight international markets. With a total of only eight assets, implementing the theory is feasible.

However, suppose that we were considering selecting a portfolio from all 500 stocks on the S&P index. Or, even worse, consider all 3000 stocks on the Russell index. Ugh! There would be $(3000)(2999)/2 \approx 4.5$ million covariances to estimate. Moreover, $\mathbf{\Omega}$ would be 3000 by 3000 and its inverse is required. However, the most serious difficulty would not be the computations. It would be data collection.

Porfolio theory was an important theoretical development; Markowitz was awarded the Nobel Prize in economics for this work. However, a practical version of this theory awaited the work of Sharpe and Lintner. Sharpe, who was Markowitz's PhD student, shared the Nobel Prize with Markowitz.

Sharpe's CAPM asserts that the tangency portfolio is also the market portfolio. This is a tremendous simplification.

5.8 Example—Global Asset Allocation

This example is taken from *Efficient Asset Management* by Richard O. Michaud. The problem is to allocate capital to eight major classes of assets: U.S. stocks, U.S. government and corporate bonds, Euros, and the Canadian, French, German, Japanese, and U.K. equity market. The historic data used to estimate expected returns, variances, and covariances consisted of 216 months (Jan 1978 to Dec 1995) of index total returns in U.S. dollars for all eight asset classes and for U.S. 30-day T-bills.

The efficient frontier, with all weights constrained to be non-negative, was found by quadratic programming and is shown in Figure 5.12. There are three reference portfolios. Michaud states that

> The index portfolio is roughly consistent with a capitalization weighted portfolio relative to a world equity benchmark for the six equity markets. The current portfolio represent a typical U.S.-based investor's global portfolio asset allocation. ... An equal weighted portfolio is useful as a reference point.

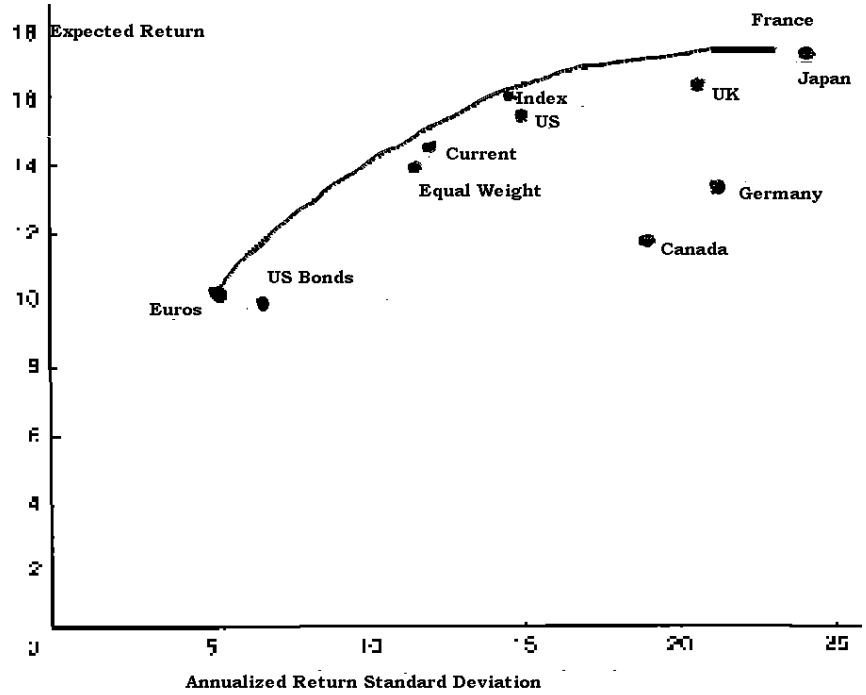


Figure 5.12: *Efficient frontier for the global asset allocation problem.*

5.9 Quadratic programming

Quadratic programming can be used to solve problems such as minimizing over x

$$\frac{1}{2}\boldsymbol{x}^{\mathsf{T}}\boldsymbol{H}\boldsymbol{x} + \boldsymbol{f}^{\mathsf{T}}\boldsymbol{x} \tag{5.26}$$

subject to

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \tag{5.27}$$

and

$$\boldsymbol{A}_{eq}\boldsymbol{x} = \boldsymbol{b}_{eq}. \tag{5.28}$$

Here for some N, $\boldsymbol{x}$ and $\boldsymbol{f}$ are $N \times 1$ vectors and $\boldsymbol{H}$ is an $N \times N$ matrix. Also, $\boldsymbol{A}$ is $m \times N$ and $\boldsymbol{b}$ is $m \times 1$ for some m, while $\boldsymbol{A}_{eq}$ is $n \times N$ and $\boldsymbol{b}_{eq}$ is $n \times 1$ for some n. In other words, quadratic programming minimizes the quadratic objective function (5.26) subject linear inequality constraints (5.27) and linear equality constraints (5.28).

We can impose nonnegativity constraints on the weights of a portfolio by solving the minimization problem above with $\boldsymbol{x} = \boldsymbol{\omega}$, $\boldsymbol{H} = \boldsymbol{\Omega}$, $\boldsymbol{f}$ equal to a $N \times 1$ vector of zeros, $\boldsymbol{A} = -\boldsymbol{I}$ (the $N \times N$ identity matrix), $\boldsymbol{b}$ equal to a $N \times 1$ vector of zeros,

$$\boldsymbol{A}_{eq} = \begin{pmatrix} \mathbf{1}^{\mathsf{T}} \\ \boldsymbol{\mu}^{\mathsf{T}} \end{pmatrix},$$

and

$$\boldsymbol{b}_{eq} = \begin{pmatrix} 1 \\ \mu_P \end{pmatrix}.$$

Then (5.26) becomes one-half the variance of the portfolio's return $\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\Omega}\boldsymbol{\omega}/2$, that is, one-half the objective function (5.2), (5.27) becomes the nonnegative constraints $\omega \geq \mathbf{0}$, and (5.28) becomes

$$\begin{pmatrix} \mathbf{1}^{\mathsf{T}} \\ \mu^{\mathsf{T}} \end{pmatrix} \boldsymbol{\omega} = \begin{pmatrix} 1 \\ \mu_P \end{pmatrix}.$$

which is the same as constraints (5.3) and (5.4). Thus, we are solving the same minimization problem as in Section 5.6.1 but with the constraint that all of the ω_i be non-negative.

The minimization problem in Section 5.6.1 can be solved by quadratic program by removing the non-negativity constraint. This is done by re-defining $\boldsymbol{A}$ to be a vector of zeros and $\boldsymbol{b}$ to be 0. With these choices of $\boldsymbol{A}$ and $\boldsymbol{b}$, constraint (5.27) is that $0\,\omega_i \leq 0$ for all i, which obviously is always satisfied and so has no effect.[4]

[4]The only reason we use a constraint that has no effect is the MATLAB's QUADPROG is written so that some constraint must be input.

Here is the documentation for MATLAB's "quadprog" illustrating several ways that this program can be used. In our applications, e.g., in the program "portfolio02QP.m," we call the program "quadprog" with a command of the type "X=QUADPROG(H,f,A,b,Aeq,beq)". This can be seen in the listing of "portfolio02QP.m" which is given later.

```
   QUADPROG Quadratic programming.  X=QUADPROG(H,f,A,b) solves the quadratic
programming problem:

 min 0.5*x'*H*x + f'*x   subject to:  A*x <= b
  x

X=QUADPROG(H,f,A,b,Aeq,beq) solves the problem above while additionally satisfying
the equality constraints Aeq*x = beq.
   X=QUADPROG(H,f,A,b,Aeq,beq,LB,UB) defines a set of lower and upper bounds
on the design variables, X, so that the solution is in the range LB <= X <=
UB. Use empty matrices for LB and UB if no bounds exist.  Set LB(i) = -Inf if
X(i) is unbounded below; set UB(i) = Inf if X(i) is unbounded above.
   X=QUADPROG(H,f,A,b,Aeq,beq,LB,UB,X0) sets the starting point to X0.
   X=QUADPROG(H,f,A,b,Aeq,beq,LB,UB,X0,OPTIONS) minimizes with the default optimization
parameters replaced by values in the structure OPTIONS, an argument created
with the OPTIMSET function.  See OPTIMSET for details.  Used options are Display,
Diagnostics, TolX, TolFun, HessMult, LargeScale, MaxIter, PrecondBandWidth,
TypicalX, TolPCG, and MaxPCGIter.  Currently, only 'final' and 'off' are valid
values for the parameter Display ('iter' is not available).
   X=QUADPROG(Hinfo,f,A,b,Aeq,beq,LB,UB,X0,OPTIONS,P1,P2,...)  passes the problem-dependent
parameters P1,P2,...  directly to the HMFUN function when OPTIMSET('HessMult',HMFUN)
is set.  HMFUN is provided by the user.  Pass empty matrices for A, b, Aeq,
beq, LB, UB, XO, OPTIONS, to use the default values.
   [X,FVAL]=QUADPROG(H,f,A,b) returns the value of the objective function at
X: FVAL = 0.5*X'*H*X + f'*X.
   [X,FVAL,EXITFLAG] = QUADPROG(H,f,A,b) returns a string EXITFLAG that describes
the exit condition of QUADPROG. If EXITFLAG is:  > 0 then QUADPROG converged
with a solution X. 0 then the maximum number of iterations was exceeded (only
occurs with large-scale method).  < 0 then the problem is unbounded, infeasible,
or QUADPROG failed to converge with a solution X.
   [X,FVAL,EXITFLAG,OUTPUT] = QUADPROG(H,f,A,b) returns a structure OUTPUT with
the number of iterations taken in OUTPUT.iterations, the type of algorithm used
in OUTPUT.algorithm, the number of conjugate gradient iterations (if used) in
OUTPUT.cgiterations, and a measure of first order optimality (if used) in OUPUT.firstorderopt.
   [X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=QUADPROG(H,f,A,b) returns the set of Lagrangian
multipliers LAMBDA, at the solution:  LAMBDA.ineqlin for the linear inequalities
A, LAMBDA.eqlin for the linear equalities Aeq, LAMBDA.lower for LB, and LAMBDA.upper
for UB.
```

Here is the program "portfolio02QP.m":

```
% Program portfolio02QP
% Lasted changed: 3/4/02

% Input mean vector and covariance matrix of returns here

bmu = [.08;.03;.05] ;
bOmega = [ .3 .02 .01 ;
  .02 .15 .03 ;
  .01 .03 .18 ] ;
```

```
Aeq = [ones(1,3);bmu'] ;
bone = ones(length(bmu),1) ;


% Compute minimum expected return and minimum return SD
ibOmega = inv(bOmega) ; % Invert Omega - "i" means inverse
A = bone'*ibOmega*bmu ;
B = bmu'*ibOmega*bmu ;
C = bone'*ibOmega*bone ;
D = B*C - A^2 ;
bg = (B*ibOmega*bone - A*ibOmega*bmu)/D ;
bh = (C*ibOmega*bmu - A*ibOmega*bone)/D ;
gg = bg'*bOmega*bg ;
hh = bh'*bOmega*bh ;
gh = bg'*bOmega*bh ;
mumin = - gh/hh ;
sdmin = sqrt( gg * ( 1 - gh^2/(gg*hh)) ) ;

ngrid = 25 ;
muP = linspace(mumin,.08,ngrid)'  ; % Grid of muP values
sigmaP = muP ; % Set up storage
sigmaP2 = sigmaP ;
omegaP = zeros(3,ngrid) ;
omegaP2 = omegaP ;

for i = 1:ngrid ;
omegaP(:,i) = quadprog(bOmega,zeros(3,1),-eye(3),zeros(3,1),Aeq,[1;muP(i)]) ;
omegaP2(:,i) =  quadprog(bOmega,zeros(3,1),zeros(1,3),0,Aeq,[1;muP(i)]) ;

sigmaP(i) = sqrt(omegaP(:,i)'*bOmega*omegaP(:,i)) ;
sigmaP2(i) = sqrt(omegaP2(:,i)'*bOmega*omegaP2(:,i)) ;

end ;

fsize = 16 ;
figure(1)
clf
p = plot(sigmaP,muP,sigmaP2,muP,'--') ;
set(p(1),'color','red') ;
set(p(2),'color','blue') ;
set(p,'linewidth',6) ;
l=legend('no negative wts','unconstrained wts',4) ;
set(gca,'fontsize',fsize) ;
set(l,'fontsize',fsize) ;

xlabel('standard deviation of return (\sigma_P)','fontsize',fsize) ;
ylabel('expected return (\mu_P)','fontsize',fsize) ;
text(sqrt(bOmega(1,1)),bmu(1),'1','fontsize',24) ;
text(sqrt(bOmega(2,2)),bmu(2),'2','fontsize',24) ;
```

```
text(sqrt(bOmega(3,3)),bmu(3),'3','fontsize',24) ;
set(gca,'ylim',[.025,.085]) ;
grid ;

print portfolio02QP.ps -depsc ;
!mv portfolio02QP.ps ~/public_html/or473/LectNotes/portfolio02QP.ps ;


figure(2)
p2 = plot(muP,omegaP(1,:),muP,omegaP(2,:),'--',muP,omegaP(3,:),'-.') ;
set(p2,'linewidth',6) ;
set(gca,'fontsize',fsize) ;
grid ;
xlabel('\mu_P','fontsize',fsize) ;
ylabel('weight','fontsize',fsize) ;
legend('w_1','w_2','w_3',0) ;

print portfolio02_wtQP.ps -depsc ;
!mv portfolio02_wtQP.ps ~/public_html/or473/LectNotes/portfolio02_wtQP.ps ;
```

5.10 Summary

- The EMH suggests that trying to pick a few great stocks is not an effective investment strategy.
- Instead of trying to buy a few "sure winners" it is better to diversify one's holdings.
- Riskier assets have higher expected returns because investors demand a reward for assuming risk. This reward is called the "risk premium."
- If there is one risky asset and one risk-free asset, then as one increases one's percentage of holdings in the risky asset, expected return and risk both increase.
- If there are two or more risky assets, then one can remove some risk without a cost in expected returns by diversifying one's holdings of risky assets.
  - The efficient frontier is a locus of points on the (expected return, risk) plot that maximize expected return for a given level of risk or minimize risk for a given expected return.
  - The efficient frontier can be found using "efficient-set mathematics" that is based on Lagrange multipliers

- When one can invest in two or more risky assets and in a risk-free asset, then the optimal portfolios are combinations of the tangency portfolio of risky assets and the risk-free asset.
  - The tangency portfolio is on the efficient horizon and can be found by efficient-set mathematics.
  - Sharpe's ratio is the reward to risk ratio and is maximized by the tangency portfolio.
- Efficient-set mathematics does not allow one to constraint portfolio weights to be non-negative.
  - Thus, short selling is needed to implement the optimal portfolios on the efficient horizon
  - To avoid negative portfolio weights, one can use quadratic programming.
  - Quadratic programming results in a new and different efficient frontier with all portfolio weights non-negative.
    - These portfolios do not require short selling.

5.11 References

Bernstein, P., (1992), *Capital Ideas: The Improbable Origins of Modern Wall Street*, Free Press, New York.

Markowitz, H. (1959), *Portfolio Selection: Efficient Diversification of Investment*, John Wiley & Sons, New York.

Merton, R.C. (1972), An analytic derivation of the efficient Portfolio frontier, *J. of Financial and Quantitative Analysis*, 7, 1851–1872.

Chapter 6

Regression: 4/3/02

6.1 Introduction

You've seen the basics of regression in ORIE 270. This chapter will review that material and present some more advanced regression techniques that will be useful for the analysis of financial data. Regression is one of the most widely used of all statistical methods, and there is much more that one can learn. Advanced regression techniques are discussed in ORIE 476 and, especially, in ORIE 576. The latter is a seven week course devoted entirely to regression.

- available data (one response variable and p predictors):
  - Y_i = value of response variable for ith observation
  - $X_{i1}, \ldots, X_{ip}$ = values of predictor variables 1 through p for the ith observation
- goals:
  - to understand how Y is related to $X_1, \ldots, X_p$.
  - to model the conditional expectation of Y given $X_1, \ldots, X_p$
  - to predict future Y values given the corresponding values of $X_1, \ldots, X_p$

6.2 Straight Line Regression

- only one predictor variable

- model is

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

  - β_0 and β_1 are the unknown intercept and slope of the line
  - $\epsilon_1, \ldots, \epsilon_n$ are iid with mean 0 and constant variance σ^2
  - often the ϵ_i's are assumed to be normally distributed

6.2.1 Least-squares estimation

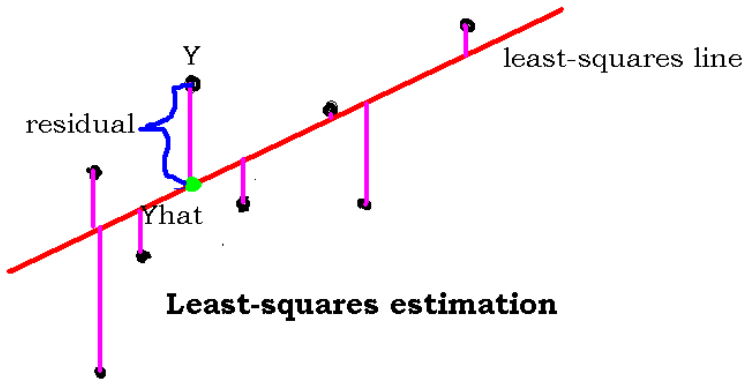


Figure 6.1: *Least-squares estimation. The magenta vertical lines are the residuals. The least-squares line is defined as the line making the sum of the squared residuals as small as possible.*

- least-squares estimate finds b_0 and b_1 to minimizes

$$\sum_{i=1}^{n} \{Y_i - (b_0 + b_1 X_i)\}^2$$

  Geometrically, we are minimizing the sum of the squared lengths of the magenta vertical lines in Figure 6.1. This lines represent the distances between the data points and the predictions using the linear

equation. The predictions themselves are called the "fitted values" or "y-hats" and the distances between the Y values and the fitted values that we are trying to minimize are called the "residuals."

- using calculus, one can show that

$$b_1 = \frac{\sum_{i=1}^n (Y_i - \overline{Y})(X_i - \overline{X})}{\sum_{i=1}^n (X_i - \overline{X})^2}.$$

and

$$b_0 = \overline{Y} - b_1 \overline{X}$$

the least-squares line is

$$\begin{aligned} \widehat{Y} &= b_0 + b_1 X = \overline{Y} + b_1 (X - \overline{X}) \\ &= \overline{Y} + \left\{ \frac{\sum_{i=1}^n (Y_i - \overline{Y})(X_i - \overline{X})}{\sum_{i=1}^n (X_i - \overline{X})^2} \right\} (X - \overline{X}) \\ &= \overline{Y} + \frac{s_{xy}}{s_x^2} (X - \overline{X}), \end{aligned}$$

where

$$s_{xy} = (n-1)^{-1} \sum_{i=1}^n (Y_i - \overline{Y})(X_i - \overline{X})$$

and s_x^2 is the sample variance of the X_i's, that is,

$$s_x^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \overline{X})^2.$$

Exercise:
Show that if $\epsilon_1, \ldots, \epsilon_n$ are IID $N(0, \sigma^2)$ then the least-squares estimates of β_0 and β_1 are also the maximum likelihood estimates.

Hint: This problem is similar to the example in Section 2.7. The only difference is that in that section, $Y_1, \ldots, Y_n$ are independent $N(\mu, \sigma^2)$, while in this exercise $Y_1, \ldots, Y_n$ are independent $N(\beta_0 + \beta_1 X_i, \sigma^2)$.

Example: Some data on weekly interest rates, from Jan 1, 1970 to Dec 31, 1993, were obtained from the Federal Reserve Bank of Chicago. The URL is:

http://www.chicagofed.org/economicresearchanddata/data/index.cfm

Figure 6.2 is a plot of changes in the 10-year Treasury rate and changes in the corporate AAA bond yield. The changes where computed using

the "difference" command in the "time series" menu of MINITAB. Missing values were created for the first observations.

The plot looks linear, so we will try linear regression using "fitted line plot" in MINITAB. The output is below and in Figure 6.3.

```
Welcome to Minitab, press F1 for help.
   1252 rows read.
Saving file as:
C:\courses\or473\minitab\WeeklyInterest.MTW

Macro is running ... please wait

Results for: WeeklyInterest.MTW

Regression Analysis: aaa_diff versus cm10_diff


The regression equation is
aaa_diff = 0.0002266 + 0.572371 cm10_diff

S = 0.0633340    R-Sq = 68.5 %    R-Sq(adj) = 68.5 %

Analysis of Variance

Source        DF       SS       MS        F      P
Regression     1  10.8950  10.8950  2716.14  0.000
Error       1249   5.0100   0.0040
Total       1250  15.9049


Fitted Line Plot: aaa_diff versus cm10_diff
```

Here is the same analysis using "regression" in MINITAB.

```
Regression Analysis: aaa_diff versus cm10_diff
```

```
The regression equation is
aaa_diff = 0.00023 + 0.572 cm10_diff

1251 cases used 1 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant     0.000227    0.001791       0.13    0.899
cm10_dif      0.57237     0.01098      52.12    0.000

S = 0.06333     R-Sq = 68.5%     R-Sq(adj) = 68.5%

Analysis of Variance

Source           DF        SS       MS        F      P
Regression        1    10.895   10.895  2716.14  0.000
Residual Error 1249     5.010    0.004
Total          1250    15.905
```

From the output we see that the least-squares estimates of the intercept and slope are .000227 and .572. The missing values created by differencing caused MINITAB to print a warning.

The remaining items of the output will be discussed shortly in Sections 6.2.2 and 6.2.3.

6.2.2 Standard errors, t-values, and p-values

Each of the coefficients in the MINITAB output has three other statistics associated with it:

- SE = standard error
  - This is estimated standard deviation of the least squares estimator and tells us the precision of that estimator.
- t-value

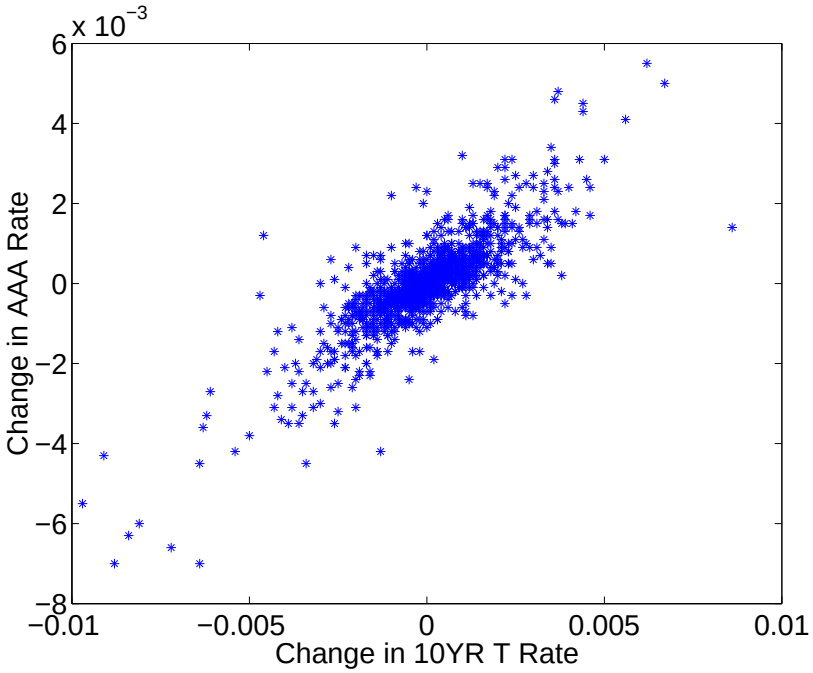


Figure 6.2: *Change in "CM10 = 10-YEAR TREASURY CONSTANT MATURITY RATE (AVERAGE, NSA)" plotted against "AAA = MOODYS SEASONED CORPORATE AAA BOND YIELDS". Data from Federal Reserve Statistical Release H.15 and available at the Chicago Federal Bank's web site.*

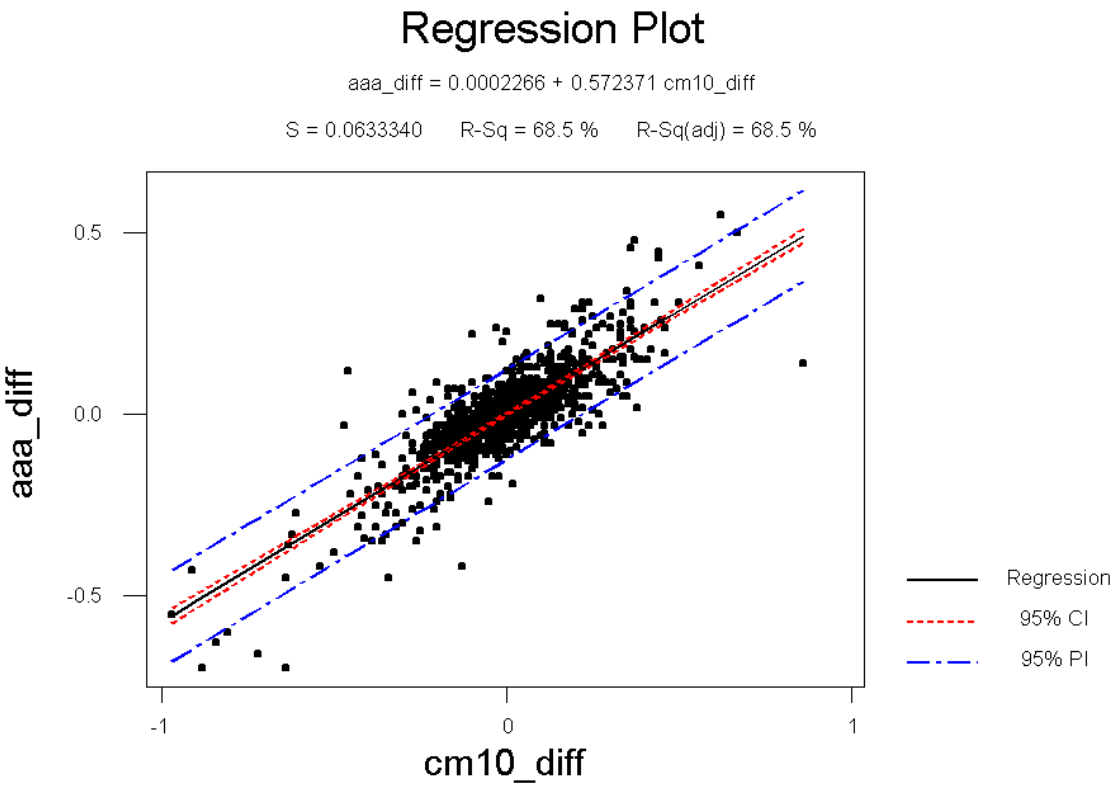


Figure 6.3: *Fitted line plot and least-squares estimates from MINITAB.*

  - This is the t-statistic for testing that the coefficient is 0.

- p-value
  - This is the p-value for the test of the null hypothesis that the coefficient is 0 versus the alternative that it is not 0.
  - The p-value is 0.000 here.

If the p-value is small as it is here, then this is evidence that the coefficient is *not* 0 which means that the predictor has some effect.

6.2.3 Analysis of variance, R^2, and F-tests

The total variation in Y can be partitioned into two parts, the variation that can be predicted by X and the part that cannot be predicted. The total variation is measured by the total sum of squares (total SS) which is

$$\text{total SS} = \sum_{i=1}^{n} (Y_i - \overline{Y})^2.$$

The variation that can be predicted is measured by regression sum of squares which is

$$\text{regression SS} = \sum_{i=1}^{n} (\widehat{Y}_i - \overline{Y})^2.$$

Finally, the amount of variation in Y that cannot be predicted by a linear function of X is measured by the residual error sum of squares which is

$$\text{residual error SS} = \sum_{i=1}^{n} (Y_i - \widehat{Y}_i)^2.$$

It can be shown algebraically that

$$\text{total SS} = \text{regression SS} + \text{residual error SS}.$$

R-squared, denoted by R^2 is

$$R^2 = \frac{\text{regression SS}}{\text{total SS}} = 1 - \frac{\text{residual error SS}}{\text{total SS}}$$

and measures the proportion of the total variation in Y that can be linearly predicted by X.

There are degrees of freedom (DF) associated with each of these "sources" of variation. The degrees of freedom for regression is p, which is the number of predictor variable. Note that p is 1 for straight-line regression. The total degrees of freedom is $n - 1$. The residual error degrees of freedom is the total degrees of freedom minus the regression degrees of freedom, which is $n - p - 1$.

The mean sum of squares (MS) for any source is its sum of squared divided by its degrees of freedom. The residual MS is an unbiased estimator of σ^2. The other means sum of squares are used for testing.

The regression MS divided by the residual error MS is called the F-statistic, or just F. The F-statistic is used to perform the so-called F-test of the null hypothesis that there is no linear relationship between any of the predictors and Y.

The entry in the column labele "P" is the p-value of this test. In our example, the p-value is 0.000 which is very strong evidence against the null hypothesis. We conclude that there *is* a relationship between changes in CM10 and changes in AAA.

6.2.4 Regression and best linear prediction

- Note the similarity between the best linear predictor

$$\widehat{Y} = E(Y) + \frac{\sigma_{xy}}{\sigma_x^2}\{X - E(X)\},$$

  and the least-squares line

$$\widehat{Y} = \overline{Y} + \frac{s_{xy}}{s_x^2}(X - \overline{X}),$$

- The least-squares line is a sample version of the best linear predictor.
- ρ_{XY}^2, the squared correlation between X and Y, is the fraction of variation in Y that can be predicted using the linear predictor.
- The sample version of ρ_{XY}^2 is R^2.

6.3 Multiple Linear Regression

The multiple regression model is

$$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \epsilon_i.$$

β_0 is the intercept. It is the expected value of Y_i when all the Xij's are zero. The regression coefficients $\beta_1, \ldots, \beta_p$ are the slopes. More precisely, β_j is the partial derivative of the expected response with respect to the jth predictor:

$$\beta_j = \frac{\partial E(Y_i)}{\partial X_{ij}}.$$

In other words, β_j is the change in the expected value of Y_i when X_{ij} changes one unit.

All coefficients are estimated by least-squares.

As an example, we will continue the to analyze weekly interest rates data but now with the 30-year Treasury rates as a second predictor. Thus $p = 2$. Here is the analysis using "regression" in MINITAB.

```
Regression Analysis: aaa_diff versus cm10_diff, cm30_diff


The regression equation is
aaa_diff = 0.00005 + 0.551 cm10_diff + 0.0284 cm30_diff

1251 cases used 1 cases contain missing values

Predictor        Coef     SE Coef          T        P
Constant     0.000047    0.001783       0.03    0.979
cm10_dif      0.55059     0.01252      43.99    0.000
cm30_dif     0.028358    0.007936       3.57    0.000

S = 0.06304     R-Sq = 68.8%     R-Sq(adj) = 68.8%

Analysis of Variance

Source            DF       SS      MS        F       P
Regression         2  10.9457  5.4728  1377.25   0.000
Residual Error  1248   4.9592  0.0040
Total           1250  15.9049

Source       DF      Seq SS
cm10_dif      1     10.8950
cm30_dif      1      0.0507
```

Note that the value of R^2 is only slightly greater than the value 68.5%

obtained when using the 10-year Treasury rate as a predictor. Both predictors have very small p-values indicating that they are statistically significant, but since there are 1252 data points, a very small effect of no practical significance can be statistically significant. This may very well be what is happening here.

6.4 Model Selection

Model selection means the selection of the predictor variables to use in the prediction equation. There are two principles to balance:

- Larger models, that is, models with more predictor variables, will have less bias and they *would* give the best predictions *if* all coefficients could be estimated without error.
- When unknown coefficents are replaced by estimates, then the prediction become less accurate, and this effect is worse when there are more coefficients to estimate.

Thus, larger models have:

- less bias
- more variability

One should not use automatic model selection software blindly. Instead, one must use common sense and knowledge of the subject matter. Nonetheless, model selection software can be used as a guide to what models one might consider.

R^2 is not a useful statistic for comparing models of different sizes. It is biased towards large models since it only looks at bias, not variance, and does not compensate for the effects of estimation error. In effect, it always choses the largest model since that model would be best if there were no estimation errors.

Unlike the ordinary R^2 value, the adjusted R^2 statistic does compensate for the effects of estimation error and the adjusted R^2 *can* be used to select models. In effect, R^2 is based on biased estimators of σ^2 but the adjusted R^2 is based on unbiased estimators.

C_p is a statistic that estimates how well a model will predict. Like the adjusted R^2 value, C_P is adjusted for the effect of estimation error. C_p is closely related to the AIC statistic used to compare time series models.

MINITAB's "best regression" command will compute ALL possible models and compare models of the same size by R^2, C_p, and adjusted R^2.[1] Because there are potentially millions of models, "best regression" doesn't print out the results for all possible models. Rather for each "size of model" (number of predictors) it prints results for a specified number of best models. The specified number is selected by the "Models of each size to print" option — the default value is 2 meaning that the two best models of each size are printed. MINITAB does not compare models of different sizes — you get to do that for yourself when looking at the output.

Here is the result of using MINITAB's "best regression" command on the interest rates data. The best model by C_p is the two-variable model. The model with just CM10 is a very close second and in practical terms is probably just as good as the two-variable model. The model using only CM30 is decidely inferior to the other models.

```
Response is aaa_diff

1251 cases used 1 cases contain missing values.
```

Vars	R-Sq	R-Sq(adj)	C-p	S	cm10_diff	cm30_diff
1	68.5	68.5	13.8	0.063334	X	
1	20.5	20.4	1936.3	0.10064		X
2	68.8	68.8	3.0	0.063038	X	X

Now let's add more potential predictor variable to make things a bit more interesting. In particular, we will add ff_diff which is the weekly change in the Federal funds rate and prime_diff which is the weekly change in the prime rate. CM10_diff and CM30_diff will continue to used as predictors as well.

[1] When comparing models of the *same* size, the rankings by R^2, adjusted R^2 and C_p are identical. Remember that "best" means larger R^2 or adjusted R^2 value or smallest C_p value.

Here are the results from using "best regression." The "Models of each size to print" option was set equal to 3. The best model according to C_p uses two predictors, CM10 and CM30. Adding either or both of the other predictors doesn't increase C_p by much (to no more than 5), but deleting either or both of CM10 or CM30 does increase C_p to at least 12.8.

```
Response is aaa_diff

1251 cases used 1 cases contain missing values.
```

Vars	R-Sq	R-Sq(adj)	C-p	S	cm10_diff	cm30_diff	prime_diff	ffr_diff
1	68.5	68.5	12.8	0.063334	X			
1	20.5	20.4	1934.0	0.10064		X		
1	5.6	5.5	2529.9	0.10966				X
2	68.8	68.8	2.1	0.063038	X	X		
2	68.5	68.5	14.0	0.063337	X			X
2	68.5	68.5	14.4	0.063348	X		X	
3	68.8	68.8	3.2	0.063040	X	X		X
3	68.8	68.8	3.6	0.063052	X	X	X	
3	68.5	68.5	15.8	0.063358	X		X	X
4	68.8	68.7	5.0	0.063061	X	X	X	X

6.5 Nonlinear Regression

Often we can derive a theoretical model relating a predictor variable and a response, but the model we derive is not linear.

For example, consider the price of par $1,000 zero-coupon bonds issued by a particular borrower, perhaps the Federal government or a certain corporation. "Zero-coupuon" means that the owner of such a bond will be paid $1,000 at the time of maturity of that bond but receives no payments prior to maturity. The price of a zero-coupon bond will always be less than par, since the par value equals the repayment of principle plus interest.

Suppose that there are a variety of bonds with different maturities and that the ith type of bond has maturity T_i. Suppose also that all market participants of zero-coupon bonds agree that the bonds should pay interest

at a continuously compounded rate r from now until the maturity of all bonds.[2]

Under this assumption, the present price of a bond with maturity T_i is

$$P_i = 1{,}000 \exp(-rT_i).$$

There will be some random variation in the observed prices. One reason for this variation is that the price of a bond can only be determined by the sale of the bond, so the observed prices have not been determined simultaneously. Instead, each bond's price was determined at the time of the last sale of a bond of that maturity. Thus, we use the regression model

$$P_i = 1{,}000 \exp(-rT_i) + \epsilon_i.$$

An estimate of r can be determined by least squares, that is, by minimizing over r the sum of squares

$$\sum_{i=1}^{n} \left\{ P_i - 1{,}000 \exp(-rT_i) \right\}^2.$$

The least-squares estimator is denoted by $\widehat{r}$.

Because the model is nonlinear, finding the least-squares estimate requires solving nonlinear equations. Fortunately, "industrial strength" statistical software such as SAS has routines for nonlinear least-squares estimation. This means that much of the difficult work has already been done for us. However, we do need to write an equation that tells the software what model we are using. In contrast, when using linear regression only the predictor variables need to be specified since the software knows that the model is linear.

Example:

This example uses simulated data with $r = .06$. The data are listed in Table 6.1.

The data and the predicted price curve using nonlinear regression are shown in Figure 6.4.

The following is the SAS program for the nonlinear regression. The "model" statement specifies the nonlinear equation that defines the model.

[2]This is not a particularly realistic assumption and will only be used to keep this example simple. In Chapter 10 we will consider more realistic models.

Maturity	Price
0.75	967.26
2.5	834.21
3	810.52
5	769.30
7.5	656.64
7.9	639.71
8	604.61
12	502.11
16	393.38

Table 6.1: Data for nonlinear regression example of zero-coupon bond prices.

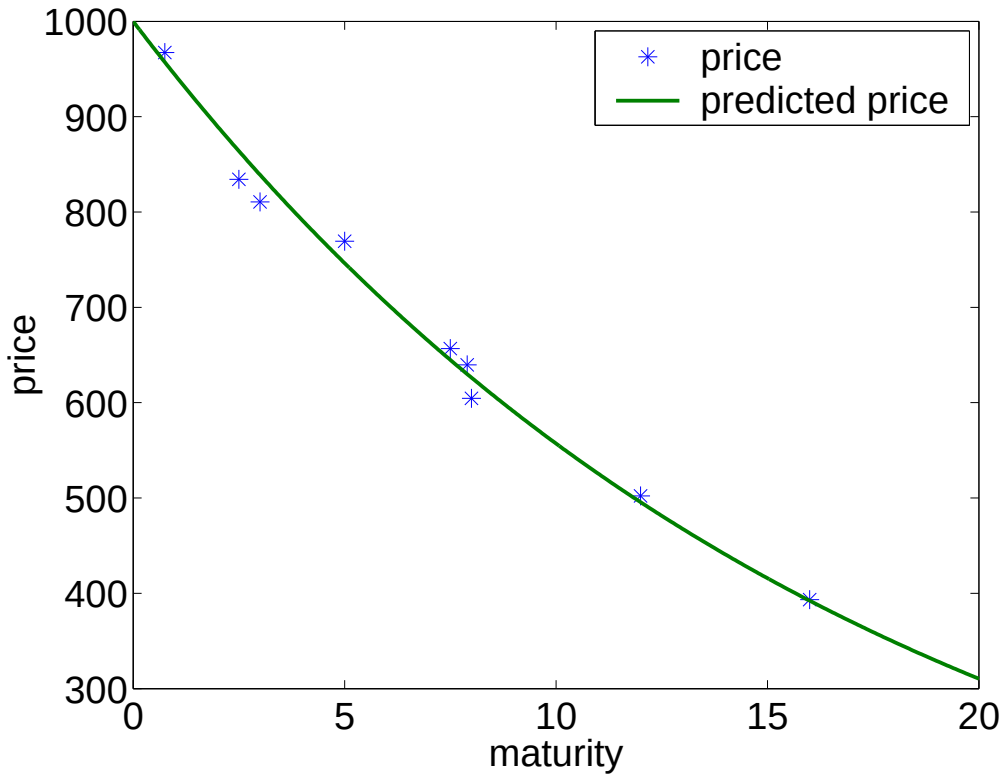


Figure 6.4: *Plot of bond prices against maturities with the predicted price from the nonlinear least-squares fit.*

Notice that "r" which is the interest rate appears in the models statement. In the previous "parm" statement, "r" is specified to be a parameter. The values "r=.02 to .09 by .005" are trial values of r.

SAS computes the sum of squares at each trial value and uses the trial value that minimizes the sum of squares as the starting value for a iterative minimization method called Gauss-Newton that is much like Newton's method.

```
options linesize = 64 ;
data bondprices ;
infile 'c:\courses\or473\data\bondprices.dat' ;
input maturity price ;
run ;
title 'Nonlinear regression using simulated zero-coupon bond data';
proc nlin ;
parm r=.02 to .09 by .005 ;
model price = 1000*exp(-r*maturity) ;
run ;
```

Here is the SAS output.

The first page of the SAS output shows the initial search of a minimum on the grid "r=.02 to .09 by .005." The sum of squares is minimized on this grid at r=.06. This is the starting value of the Gauss-Newton search.

The second page of the output shows the results of the Gauss-Newton search for the least-squares estimate. The Gauss-Newton method converged very quickly.

Later on page 2 of the output[3], there is an F-test that r is zero. In this context, the null hypothesis is a little silly and we shouldn't be surprised that the p-value is < 0.0001. The least-squares estimate is $\widehat{r} = .0585$ with a standard error of $.00149$. An approximate 95% confidence interval is (0.0551, 0.0619). The confidence interval does contain the true value of r, .06.

```
Nonlinear regression using simulated zero-coupon bond data  1
                            12:50 Thursday, January 31, 2002

                        The NLIN Procedure
                           Grid Search
                      Dependent Variable price
```

[3]Page 2 of the output stretches across two pages of these notes.

r	Sum of Squares
0.0200	390066
0.0250	279853
0.0300	192505
0.0350	124990
0.0400	74665.1
0.0450	39230.4
0.0500	16679.7
0.0550	5263.9
0.0600	3456.9
0.0650	9926.6
0.0700	23509.9
0.0750	43191.1
0.0800	68082.5
0.0850	97408.6
0.0900	130491

```
Nonlinear regression using simulated zero-coupon bond data  2
                            12:50 Thursday, January 31, 2002

                        The NLIN Procedure
                         Iterative Phase
                    Dependent Variable price
                      Method: Gauss-Newton

                                      Sum of
                Iter           r     Squares

                   0      0.0600      3456.9
                   1      0.0585      3072.8
                   2      0.0585      3072.8

NOTE: Convergence criterion met.

                        Estimation Summary

                Method                 Gauss-Newton
                Iterations                        2
                R                          4.889E-6
                PPC(r)                     3.518E-7
                RPC(r)                     0.000104
                Object                     2.084E-6
                Objective                  3072.809
                Observations Read                 9
                Observations Used                 9
                Observations Missing              0

      NOTE: An intercept was not specified for this model.
```

Source	DF	Sum of Squares	Mean Square	F Value	Approx Pr > F
Regression	1	4490011	4490011	11689.7	<.0001
Residual	8	3072.8	384.1		
Uncorrected Total	9	4493084			
Corrected Total	8	252587			

Parameter	Estimate	Approx Std Error	Approximate 95% Confidence Limits	
r	0.0585	0.00149	0.0551	0.0619

6.6 Summary

- available data: one response variable and p predictors
- Y_i = value of response variable for ith observation
- $X_{i1}, \ldots, X_{ip}$ = values of predictor variables 1 through p for the ith observation
- straight Line Regression model is

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- least-squares estimate finds b_0 and b_1 to minimizes

$$\sum_{i=1}^{n} \{Y_i - (b_0 + b_1 X_i)\}^2$$

- SE = standard error
  - This is estimated standard deviation of the least squares estimator and tells us the precision of that estimator.
- t-value
  - This is the t-statistic for testing that the coefficient is 0.

- p-value
  - This is the p-value for the test of the null hypothesis that the coefficient is 0 versus the alternative that it is not 0.
- $$\text{total SS} = \text{regression SS} + \text{residual error SS}.$$
- R-squared, denoted by R^2, is
$$R^2 = \frac{\text{regression SS}}{\text{total SS}} = 1 - \frac{\text{residual error SS}}{\text{total SS}}$$
- The regression MS divided by the residual error MS is called the F-statistic, or just F.
- The F-statistic is used to perform the so-called F-test of the null hypothesis that there is no linear relationship between any of the predictors and Y.
- The multiple regression model is
$$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \epsilon_i.$$
- Larger models, that is, models with more predictor variables, will have less bias and they *would* give the best predictions *if* all coefficients could be estimated without error.
- When unknown coefficents are replaced by estimates, then the prediction become less accurate
  - this effect is worse when there are more coefficients to estimate
- One should not use automatic model selection software blindly
- R^2 is not a useful statistic for comparing models of different sizes
- the adjusted R^2 statistic does compensate for the effects of estimation error and the adjusted R^2 *can* be used to select models.
- C_p is a statistic that estimates how well a model will predict.
- MINITAB's "best regression" command will compute ALL possible models and compare models of the same size by R^2, C_p, and adjusted R^2.

6.7 References

Neter, J., Kutner, M.H., Nachtsheim, C.J., and Wasserman, W., (1996), *Applied Linear Statistical Models, 4th ed.*, Irwin, Chicago. [Good introduction to regression and other linear models.]

Chapter 7

The Capital Asset Pricing Model: 5/15/02

7.1 Introduction to CAPM

The CAPM (capital asset pricing model has a variety of uses:

- It provides a theoretical justification for the widespread practice of "passive" investing known as *indexing*.
  - Indexing means holding a diversified portfolio in which securities are held in the same relative proportions as in a broad market index such as the S&P 500. Individual investors can do this easily by holding shares in an *index fund*.
- CAPM can provide estimates of expected rates of return on individual investments
- CAPM can establish "fair" rates of return on invested capital in regulated firms or in firms working on a cost-plus basis — what should the "plus" be?

CAPM starts with the question, what would be the risk premiums on securites if the following assumptions were true?

- The market prices are "in equilibrium."
  - In partcular, for each asset, supply equals demand.
- Everyone has the same forecasts of expected returns and risks.

- All investors chose portfolios optimally according to the priniciples of efficient diversification discussed in Chapter 5.
  - This implies that everyone holds the tangency portfolio of risky assets as well as the risk-free asset.
    - only the mix of the tangency portfolio and the risk-free varies between investors
- The market rewards people for assuming unavoidable risk, but there is no reward for needless risks due to inefficient portfolio selection.
  - Therefore, the risk-premium on a single security is not due to its "stand alone" risk, but rather to its contribution to the risk of the tangency portfolio.
    - The various components of risk will be discussed in Section 7.4.

As in Chapter 5, "return" can either refer to one-period net returns or one-period log returns.

Suppose that there are exactly three assets with a total market value of $100 billion.

- Stock A: $60 billion
- Stock B: $30 billion
- risk-free: $10 billion

The market portfolio of Stock A to Stock B is 2:1. CAPM says that under equilibrium, all investors will hold Stock A to Stock B in a 2:1 ratio. Therefore, the tangency portfolio puts weight 2/3 on Stock A and 1/3 on Stock B and *all* investors will have two-thirds of their allocation to risky assets in Stock A and one-third in Stock B.

Suppose there was too little of Stock A and too much of Stock B for everyone to have a 2:1 allocation. For example, suppose that there were one million shares of each stock and the price per share was $60 for Stock A and $40 for Stock B. Then the market portfolio must hold Stock A to Stock B in a 3:2 ratio, not 2:1. Not everyone could hold the tangency portfolio, though everyone would want to. Thus, prices would be in disequilibrium and would change. The price of Stock A would go up since the supply of Stock A is less than the demand. Similarly the price of Stock B would

go down. As these prices changed, so would expected returns and the tangency portfolio would change.

These changes in prices and expected returns would stop when the market portfolio was equal to the tangency portfolio, so that prices were in equilibrium. At least, this adjustment to equilibrium would happen under the ideal conditions of economic theory. The real world would be a little messier. The underlying message from theory, is however, correct. Prices adjusts as all investors look for an efficient portfolio and supply and demand converge to each other.

The market portfolio is 9:1 risky to risk-free. In total, investors must hold risky to risk-free in a 9:1 ratio — they *are* the market.

For an individual investor, the risky:risk-free ratio will depend on that investor's risk aversion.

- At one extreme, a portfolio of all risk-free has a standard deviation of returns equal to 0.
- At the other extreme, all risky assets, the standard deviation is maximized. [1]

At equilibrium, returns on risky and risk-free assets are such that aggregate demand for risk-free assets equals supply.

7.2 The capital market line (CML)

The capital market line (CML) relates the excess expected return on an efficient portfolio to its risk; "excess expected return" means the amount by which the expected return exceeds the risk-free rate of return. The CML is

$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R, \tag{7.1}$$

where R is the return on a given efficient portfolio (mixture of the market portfolio[2] and the risk-free asset), $\mu_R = E(R)$, μ_f is the rate of return on the risk-free asset, R_M is the return on the market portfolio, $\mu_P = E(R_M)$, σ_M is the standard deviation of the return on the market portfolio, and σ_R is the standard deviation of return on the portfolio.

[1] This assumes no buying on margin. If we allow negative positions in the risk-free, then there is no limit to the risk.

[2] Remember that the market portfolio is assumed to be the same as the tangency portfolio.

In (7.1) μ_f, μ_M, and σ_M are constant. What varies is σ_R and μ_R. These vary as we change the efficient portfolio R. Think of the CML as showing how μ_R depends on σ_R.

The slope of the CML is, of course,

$$\frac{\mu_M - \mu_f}{\sigma_M}$$

which can be interpreted as the ratio of the "risk premium" to the standard deviation of the market porffolio. This is Sharpe's "reward-to-risk ratio." Equation (7.1) can be rewritten as

$$\frac{\mu_R - \mu_f}{\sigma_R} = \frac{\mu_M - \mu_f}{\sigma_M},$$

which says that the reward-to-risk ratio for any efficient portfolio equals that ratio for the market portfolio.

Example: Suppose that risk-free rate of interest is $\mu_f = 0.06$, that the expected return on the market portfolio is $\mu_M = .15$, and the risk of the market portfolio is $\sigma_M = 0.22$. Then the slope of the CML is $(.15 - .06)/.22 = 9/22$. The CML of this example is illustrated in Figure 7.1.

The CML is easy to derive. Consider an efficient portfolio that allocates a proportion w of its assets to the market portfolio and $(1 - w)$ to the risk-free asset. Then

$$R = wR_M + (1 - w)\mu_f = \mu_f + w(R_M - \mu_f). \tag{7.2}$$

Therefore, taking expectations in (7.2)

$$\mu_R = \mu_f + w(\mu_M - \mu_f). \tag{7.3}$$

Also from (7.2)

$$\sigma_R = w\sigma_M,$$

or

$$w = \frac{\sigma_R}{\sigma_M}. \tag{7.4}$$

Substituting (7.4) into (7.3) gives the CML.

CAPM says that the optimal way to invest is to:

1. Decide on the risk σ_R that you can tolerate, $0 \leq \sigma_R \leq \sigma_M$.[3]
2. Calculate $w = \sigma_R/\sigma_M$.
3. Invest w proportion of your investment in an index fund, i.e., a fund that tracks the index.
4. Invest $1 - w$ proportion of your investment in risk-free treasury bills, or a money-market fund that invests in T-bills.

Alternatively,

1. Choose the reward $\mu_R - \mu_f$ that you want. The only constraint is that $\mu_f \leq \mu_R \leq \mu_M$ so that $0 \leq w \leq 1$ and even this constraint can be relaxed if one is permitted to buy assets on margin.
2. Calculate
$$w = \frac{\mu_R - \mu_f}{\mu_M - \mu_f}.$$
3. Do steps 3 and 4 as above.

One can view $w = \sigma_R/\sigma_M$ as is an index of the risk aversion of the investor. The smaller the value of w the more risk averse the investor. If an investor has w equal to 0, then that investor is 100% in risk-free assets. Similarly, an investor with $w = 1$ is totally invested in the tangency portfolio of risky assets.

[3] $\sigma_R > \sigma_M$ is possible by borrowing money to buy risky assets on margin.

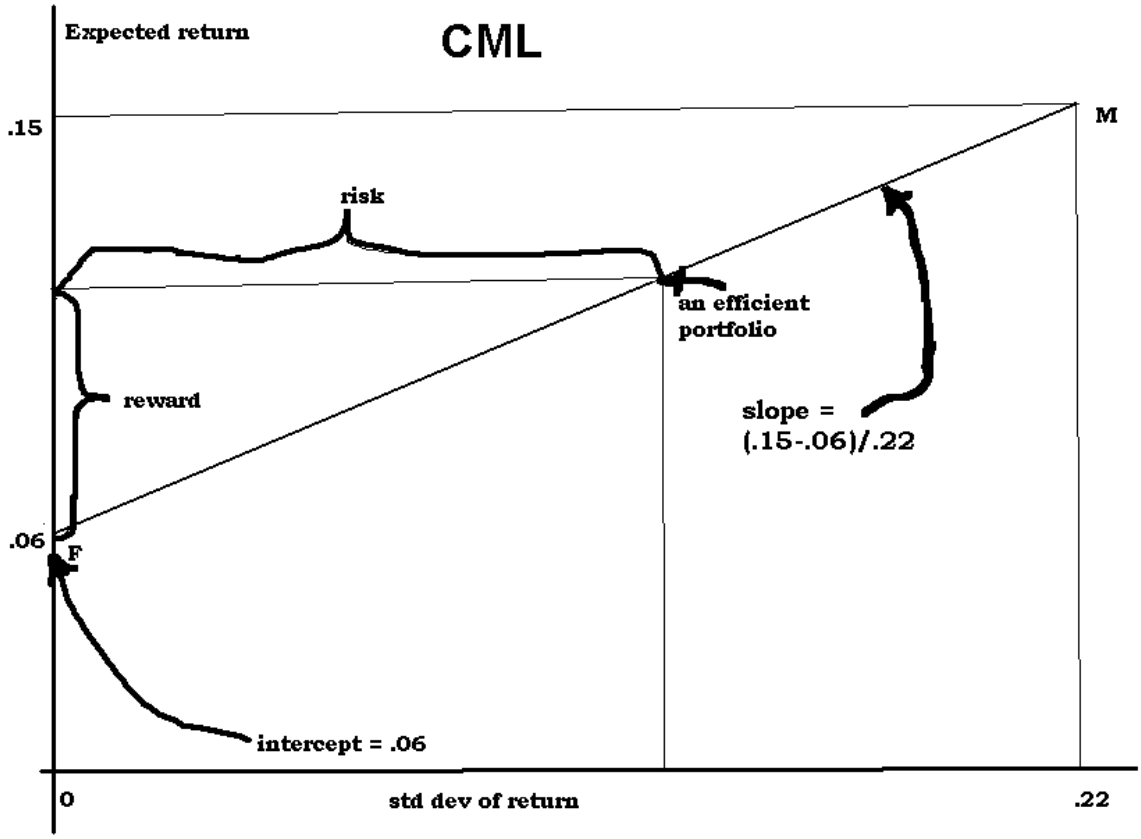


Figure 7.1: *CML when* $\mu_f = 0.06$, $\mu_M = 0.15$, *and* $\sigma_M = 0.22$. *All efficient portfolios are on the line connecting the risk-free asset (F) and the market portfolio (M). Therefore, the reward-to-risk ratio is the same for all efficient portfolios, including the market portfolio.*

7.3 Betas and the Security Market Line

The Security Market Line (SML) relates the excess return on an asset to the slope of its regression on the market portfolio.

Suppose that there are many securities indexed by j. Define

$\sigma_{jM} =$ covariance between the jth security and the market portfolio.

Also, define

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}. \tag{7.5}$$

It follows from the theory of best linear prediction in Section 2.2 that β_j is the slope of the best linear predictor of the jth security's returns based upon the returns of the market portfolio. This fact follows from equation (2.8) for the slope of a best linear prediction equation. In fact, the best linear predictor of R_j based on R_m is

$$\widehat{R}_j = \beta_{0,j} + \beta_j R_M, \tag{7.6}$$

where β_j in (7.6) is the same as in (7.5).

Another way to appreciate the significance of β_j is based on linear regression. As discussed later in Section 6.2.4, linear regression is a method for estimating the coefficients of the best linear predictor based upon data. To apply linear regression, uppose that we have a bivariate time series $(R_{jt}, R_{Mt})_{t=1}^n$ of returns on the jth asset and the market portfolio. Then, the estimated slope of the linear regression regression of R_{jt} on R_{Mt} is

$$\hat{\beta}_j = \frac{\sum_{t=1}^n (R_{jt} - \overline{R}_j)(R_{Mt} - \overline{R}_M)}{\sum_{t=1}^n (R_{Mt} - \overline{R}_M)^2}, \tag{7.7}$$

which is an estimate of σ_{jM} divided by an estimate of σ_M^2.

Let μ_j be the expected return on the jth security. Then $\mu_j - \mu_f$ is the "risk premium" (or "reward for risk" or "excess expected return") for that security. Using CAPM, it can be shown that

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f). \tag{7.8}$$

This equation, which is called the security market line (SML), will be derived in Section 7.5.2. In (7.8) β_j is a variable in the linear equation, not the slope; more precisely, μ_j is a linear function of β_j with slope $\mu_M - \mu_f$. This point is worth remembering. Otherwise, there could be some confusion since β_j was defined earlier as a slope of a regression model. In other

words, β_j is a slope in one context but is the independent variable in the SML. One can estimate β_j using (7.7) and then plug this estimate into (7.8)

The SML says that the risk premium of the jth asset is the product of its beta (β_j) and the risk premium of the market portfolio ($\mu_M - \mu_f$). β_j measures both the riskiness of the jth asset and the reward for assuming that riskiness. β_j is, therefore, a measure of how "aggressive" the jth asset is. By definition, the beta for the market portfolio is 1, i.e., $\beta_M = 1$. Therefore,

$$\begin{aligned} \beta_j > 1 \quad &\Rightarrow \text{"aggressive"} \\ \beta_j = 1 \quad &\Rightarrow \text{"average risk"} \\ \beta_j < 1 \quad &\Rightarrow \text{"not aggressive".} \end{aligned}$$

Figure 7.2 illustrated the SML and an asset, J, that is not on the SML. This asset contradicts the CAPM; according to CAPM no such asset exists.

Consider what would happen if an asset like J did exist. Investors would not want to buy it because its risk premium is too low for the risk given by its beta. They would invest less in J and more in other securities. Therefore the price of J would decline and its expected return would increase. After that increase, the asset J would be on the SML, or so the theory predicts. In other words, J is *mispriced* according to CAPM.

7.3.1 Examples of betas

Netscape's home page has a link to stock quotes from Salomon Smith Barney. If you request a quote on a stock, you will be given menu for choosing further information about the company. Under "profile" you will find the five-year beta of the company, its industry, and the S&P 500. Table 7.3.1 has some "five-year betas" that I took from this web site between February 27 and March 5, 2001. The beta for the S&P 500 is given as 1.00; why?

7.3.2 Comparison of the CML with the SML

The CML applies only to the return R of an efficient portfolio. It can be arranged so as to relate the excess expect return of that portfolio to the excess expected return of the market portfolio:

$$\mu_R - \mu_f = \left(\frac{\sigma_R}{\sigma_M}\right)(\mu_M - \mu_f). \tag{7.9}$$

Stock (symbol)	**Industry**	**Stock's β**	**Ind's β**
Celanese (CZ)	Synthetics	0.13	0.86
General Mills (GIS)	Food - major diversif	0.29	0.39
Kellogg (K)	Food - major, diversif	0.30	0.39
Proctor & Gamble (PG)	Cleaning Prod	0.35	0.40
Exxon-Mobil (XOM)	Oil/gas	0.39	0.56
7-Eleven (SE)	Grocery stores	0.55	0.38
Merck (Mrk)	Major drug manuf	0.56	0.62
McDonalds (MCD)	Restaurants	0.71	0.63
McGraw-Hill (MHP)	Pub - books	0.87	0.77
Ford (F)	Auto	0.89	1.00
Aetna (AET)	Health care plans	1.11	0.98
General Motors (GM)	Major auto manuf	1.11	1.09
AT&T (T)	Long dist carrier	1.19	1.34
General Electric (GE)	Conglomerates	1.22	0.99
Genentech (DNA)	Biotech	1.43	0.69
Microsoft (MSFT)	Software applic.	1.77	1.72
Cree (Cree)	Semicond equip	2.16	2.30
Amazon (AMZN)	Net soft & serv	2.99	2.46
Doubleclick (Dclk)	Net soft & serv	4.06	2.46

Table 7.1: Selected stocks and the industries they are in. Betas are given for each stock(Stock's β) and its industry (Ind's β).

The SML applies to *any* asset and like the CML relates its excess expected return to the excess expected return of the market portfolio:

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f). \tag{7.10}$$

If we take an efficient portfolio and consider it as an asset, then μ_R and μ_j both denote the expected return on that portfolio/asset. Both (7.9) and (7.10) hold so that

$$\frac{\sigma_R}{\sigma_M} = \beta_R.$$

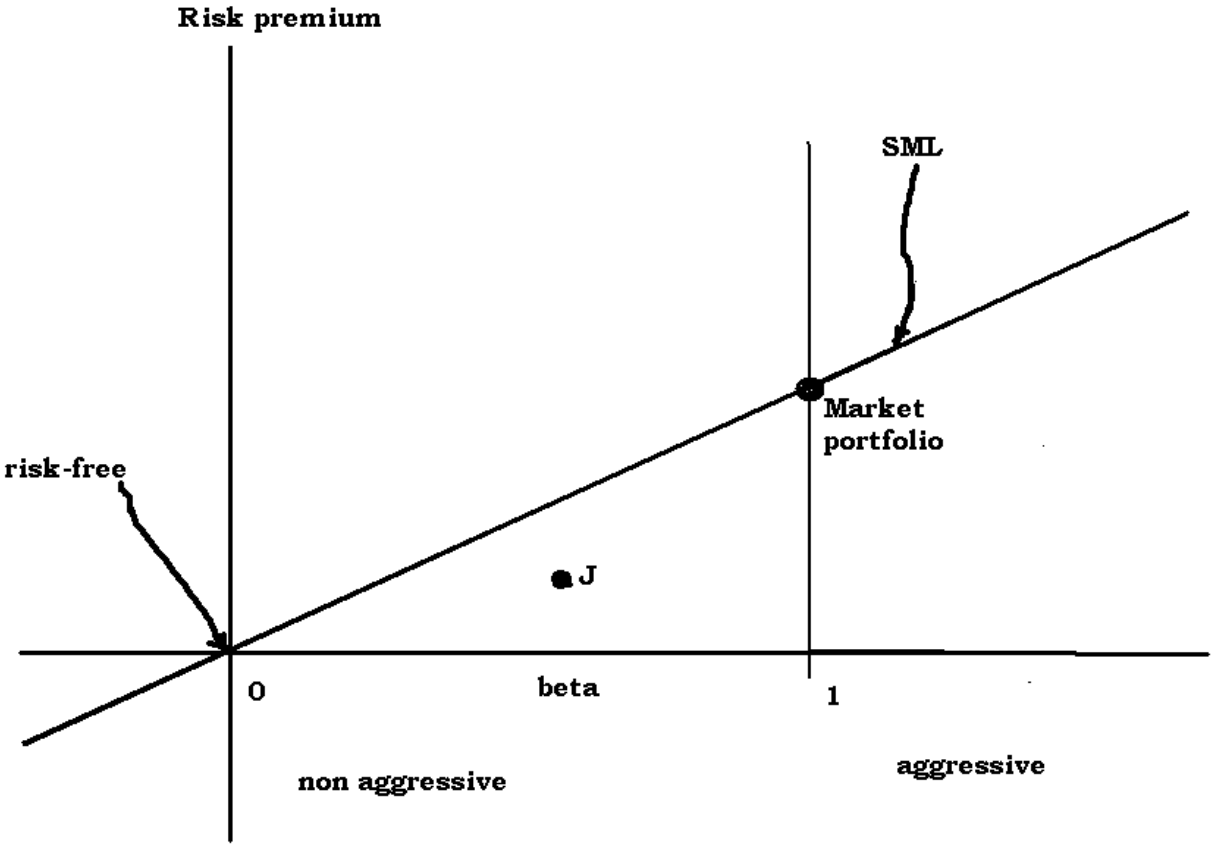


Figure 7.2: *Security market line (SML) showing that the risk premium of an asset is a linear function of the asset's beta. J is a security not on the line and a contradiction to CAPM. Theory predicts that the price of J will decrease until J is on the SML.*

7.4 The security characteristic line

Let R_{jt} be the return at time t on the jth asset. Similarly, let R_{Mt} and μ_{ft} be the return on the market portfolio and the risk-free return at time t. The security characteristic line (sometimes shortened to the characteristic line) is a regression model:

$$R_{jt} = \mu_{ft} + \beta_j(R_{Mt} - \mu_{ft}) + \epsilon_{jt}, \tag{7.11}$$

where ϵ_{jt} is $N(0, \sigma^2_{\epsilon j})$. It is often assumed that the ϵ_{jt}'s are uncorrelated across assets, that is, that ϵ_{jt} is uncorrelated with $\epsilon_{j't}$ for $j \neq j'$. This assumption has important ramifications for risk reduction by diversification; see Section 7.4.1.

Let $\mu_j = E(R_{jt})$ and $\mu_M = E(R_{Mt})$. Taking expectations in (7.11) we get

$$\mu_j = \mu_f + \beta_j(\mu_M - \mu_f),$$

which is our friend the SML again. The SML gives us information about expected returns, but not about the variance of the returns. For the latter we need the characteristic line. The characteristic line is said to be a "return generating process" since it gives us a probability model of the returns, not just a model of their expected values.

An analogy to the distinction between the SML and characteristic line is this. The regression line $E(Y|X) = \beta_0 + \beta_1 X$ gives of the expected value of Y given X but not the conditional probability distribution of Y given X. The regression model

$$Y_t = \beta_0 + \beta_1 X_t + \epsilon_t, \text{ and } \epsilon_t \sim N(0, \sigma^2)$$

does give us this conditional probability distribution.

The characteristic line implies that

$$\sigma^2_j = \beta^2_j \sigma^2_M + \sigma^2_{\epsilon j},$$

$$\sigma_{jj'} = \beta_j \beta_{j'} \sigma^2_M$$

for $j \neq j'$, and that

$$\sigma_{Mj} = \beta_j \sigma^2_M.$$

The total risk of the jth asset is

$$\sigma_j = \sqrt{\beta^2_j \sigma^2_M + \sigma^2_{\epsilon j}}.$$

The risk has two components: $\beta^2_j \sigma^2_M$ is called the market or systematic component of risk and $\sigma^2_{j\epsilon}$ is called the unique, nonmarket, or unsystematic component of risk.

7.4.1 Reducing unique risk by diversification

The market component cannot be reduced by diversification, but the unique component can be reduced in this way.

Suppose that there are N assets with returns $R_{1t}, \ldots, R_{Nt}$ for holding period t. If we form a portfolio with weights $w_1, \ldots, w_N$ then the return of the portfolio is

$$R_{Pt} = w_1 R_{1t} + \cdots + w_N R_{Nt}.$$

Let R_{Mt} be the return on the market portfolio. According to the characteristic line model $R_{jt} = \mu_{ft} + \beta_j(R_{Mt} - \mu_{ft}) + \epsilon_{jt}$ so that

$$R_{Pt} = \mu_{ft} + \left(\sum_{j=1}^{N} \beta_j w_j\right)(R_{Mt} - \mu_{Ft}) + \sum_{j=1}^{N} w_j \epsilon_{jt}.$$

Therefore, the portfolio beta is

$$\beta_P = \sum_{j=1}^{N} w_j \beta_j,$$

and the "epsilon" for the portfolio is

$$\epsilon_{Pt} = \sum_{j=1}^{N} w_j \epsilon_{jt}.$$

We will now assume that $\epsilon_{1t}, \ldots, \epsilon_{Nt}$ are uncorrelated. Therefore, by equation (2.15)

$$\sigma^2_{\epsilon P} = \sum_{j=1}^{N} w_j^2 \sigma^2_{\epsilon j}.$$

Example

Suppose that $w_j = 1/N$ for all j. Then

$$\beta_P = \frac{\sum_{j=1}^{N} \beta_j}{N},$$

and

$$\sigma^2_{\epsilon P} = \frac{N^{-1} \sum_{j=1}^{N} \sigma^2_{\epsilon j}}{N} = \frac{\overline{\sigma}^2_{\epsilon}}{N},$$

where $\overline{\sigma}_\epsilon^2$ is the average of the $\sigma_{\epsilon j}^2$. If $\sigma_{\epsilon j}^2$ is a constant, say σ_ϵ^2 for all j, then

$$\sigma_{\epsilon P} = \frac{\sigma_\epsilon}{\sqrt{N}}.$$

For example, suppose that σ_ϵ is 5%. If $N = 20$, then $\sigma_{\epsilon P}$ is 1.12%. If $N = 100$, then $\sigma_{\epsilon P}$ is 0.5%. There are approximately 1600 stocks on the NYSE; if $N = 1600$, then $\sigma_{\epsilon P} = 0.0125\%$.

Are the assumptions sensible?

A key assumption that allows nonmarket risk to be removed by diversification is that $\epsilon_{1t}, \ldots, \epsilon_{Nt}$ are uncorrelated. This assumption implies that *all* correlation among the cross-section[4] of asset returns is due to a single cause and that cause is measured by the market index. For this reason, the characteristic line is a "single factor" or "single index" model with R_{Mt} the "factor."

This assumption of uncorrelated ϵ_{jt} would not be valid if, say, two energy stocks are correlated over and beyond their correlation due to the market index. In this case, unique risk could not be eliminated by holding a large portfolio of all energy stocks. However, one can show that if there are many market sectors and the sectors are uncorrelated, then one can elminate nonmarket risk by diversifying across all sectors. All that is needed is to treat the sectors themselves as the underlying assets and then apply the CAPM theory.

7.5 Some theory

In this section we will show that σ_{jM} quantifies the contribution of the jth asset to the risk of the market portfolio. Also, we will derive the SML.

7.5.1 Contributions to the market portfolio's risk

Suppose that the market consists of N risky assets and that $w_{1M}, \ldots, w_{NM}$ are the weights of these assets in the market portfolio. Then

$$R_{Mt} = \sum_{i=1}^{N} w_{iM} R_{it},$$

[4] "Cross-section" returns means returns across assets within a *single* holding period.

which implies that the covariance between the return on the jth asset and the return on the market portfolio is

$$\sigma_{jM} = \text{Cov}\left(R_{jt}, \sum_{i=1}^{N} w_{iM} R_{it}\right) = \sum_{i=1}^{N} w_{iM}\sigma_{ij}. \tag{7.12}$$

Exercise: Show that equation (7.12) follows from equation (2.14).

Therefore,

$$\sigma_M^2 = \sum_{j=1}^{N}\sum_{i=1}^{N} w_{jM} w_{iM}\sigma_{ij} = \sum_{j=1}^{N} w_{jM}\left(\sum_{i=1}^{N} w_{iM}\sigma_{ij}\right) = \sum_{j=1}^{N} w_{jM}\sigma_{jM}. \tag{7.13}$$

Equation (7.13) shows that $w_{jM}\sigma_{jM}$ is the contribution of the jth asset to the risk of the market portfolio.

7.5.2 Derivation of the SML

This derivation was adopted from Sharpe, Alexander, and Bailey (1999). It is a nice application of calculus and geometric reasoning. It is based on a clever idea of putting together a portfolio with two assets, the market portfolio and the ith risky asset, and then looking at the locus in reward-risk space as the portfolio weight assigned to the ith risky asset varies.

Consider a portfolio P with weight w_i given to the ith risky asset and weight $(1 - w_i)$ given to the market portfolio. The return on this portfolio is

$$R_{Pt} = w_i R_{it} + (1 - w_i)R_{Mt}.$$

The expected return is

$$\mu_P = w_i\mu_i + (1 - w_i)\mu_M, \tag{7.14}$$

and the risk is

$$\sigma_P = \sqrt{w_i^2\sigma_i^2 + (1 - w_i)^2\sigma_M^2 + 2w_i(1 - w_i)\sigma_{iM}}.$$

As we vary w_i we get the locus of points on (σ, μ) space that is shown as a blue curve in Figure 7.3.

Key idea: The derivative of this locus of points evaluated at the market portfolio (which is the point where $w_i = 0$) is equal to the slope of the CML. We can calculate this derivative and equate it to the slope of the CML to see what we get. The result will be the SML.

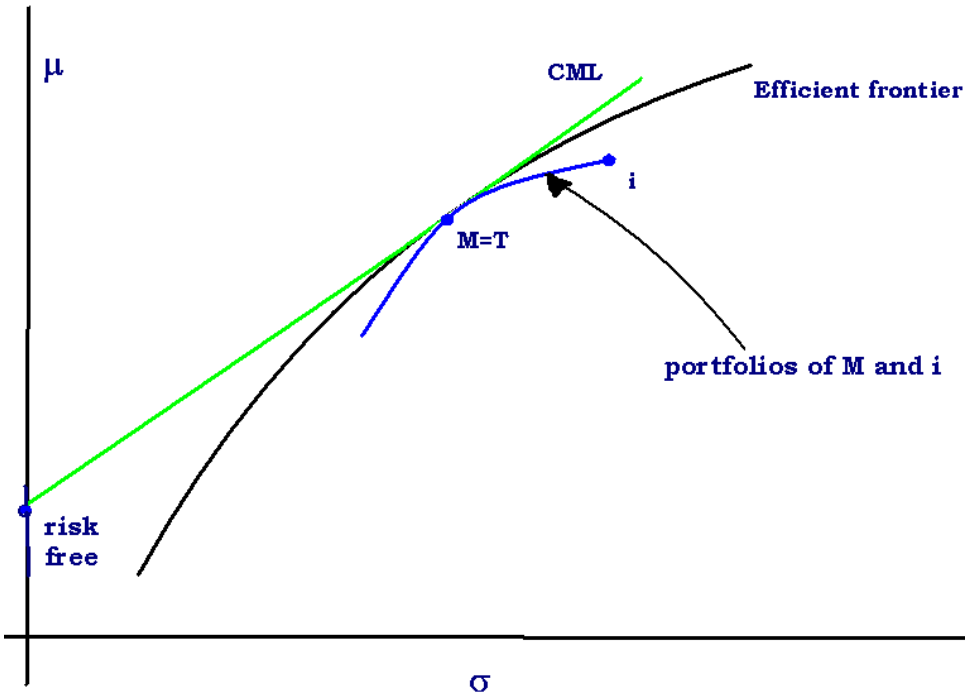


Figure 7.3: *Derivation of the SML. M is the market portfolio and T is the tangency portfolio; they are equal according to the CAPM. The blue curve is the locus of portfolios combining asset i and the market portfolio. The derivative of this curve at M is equal to the slope of the CML, since this curve is tangent to the CML at M.*

We have from (7.14)

$$\frac{d\,\mu_P}{d\,w_i} = \mu_i - \mu_M,$$

and

$$\frac{d\,\sigma_P}{d\,w_i} = \frac{1}{2}\sigma_P^{-1}\left\{2w_i\sigma_i^2 - 2(1-w_i)\sigma_M^2 + 2(1-2w_i)\sigma_{iM}\right\}.$$

Therefore,

$$\frac{d\,\mu_P}{d\,\sigma_P} = \frac{d\,\mu_P/d\,w_i}{d\,\sigma_P/d\,w_i} = \frac{(\mu_i-\mu_M)\sigma_P}{w_i\sigma_i^2 - \sigma_M^2 + w_i\sigma_M^2 + \sigma_{iM} - 2w_i\sigma_{iM}}.$$

Next,

$$\frac{d\,\mu_P}{d\,\sigma_P}\Big|_{w_i=0} = \frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}.$$

Recall that $w_i = 0$ is the point M = T in Figure 7.3 where the blue-colored locus is tangent to the CML. Therefore,

$$\frac{d\,\mu_P}{d\,\sigma_P}\Big|_{w_i=0}$$

must equal the slope of the CML which is $(\mu_M - \mu_f)/\sigma_M$. Therefore,

$$\frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\mu_M - \mu_f}{\sigma_M}.$$

which, after some algebra, gives us

$$\mu_i - \mu_f = \frac{\sigma_{iM}}{\sigma_M^2}(\mu_M - \mu_f) = \beta_i(\mu_M - \mu_f)$$

which is the SML given in equation (7.8).

7.6 Estimation of beta and testing the CAPM

Recall the security characteristic line

$$R_{jt} = \mu_{ft} + \beta_j(R_{Mt} - \mu_{ft}) + \epsilon_{jt}, \tag{7.15}$$

Let $R^*_{jt} = R_{jt} - \mu_{ft}$ be the excess return on the jth security and let $R^*_{Mt} = R_{Mt} - \mu_{ft}$ be the excess return on the market portfolio. Then (7.15) can be written as

$$R^*_{jt} = \beta_j R^*_{Mt} + \epsilon_{jt}. \tag{7.16}$$

Equation (7.16) is a regression model without an intercept and with β_j as the slope. A more elaborate model is

$$R^*_{jt} = \alpha_j + \beta_j R^*_{Mt} + \epsilon_{jt}. \tag{7.17}$$

which includes an intercept. The CAPM says that $\alpha_j = 0$ but by allowing $\alpha_j \neq 0$ we recognize the possibility of mispricing.

Given series R_{jt}, R_{Mt}, and μ_{ft} for $t = 1, \ldots, n$, we can calculate R^*_{jt} and R^*_{Mt} and regress R^*_{jt} on R^*_{Mt} to estimate α_j, β_j, and $\sigma^2_{\epsilon j}$.

By testing the null hypothesis that $\alpha_j = 0$ we are testing whether the jth asset is mispriced accoridng to the CAPM.

Here is an example done in MINITAB. Five years of monthly data, March 1996 to February 2001, were used. The raw data are in the Excel file "Datastream01.xls" on the course home page.

The excess returns on Microsoft were found by taking the logarithms of the Microsoft prices, which are the raw data in the file, and then differing the log prices. This gives the log returns for Microsoft. The 3-month T-bill rates given in the file were used as the risk-free returns.[5] The T-bill rate is an annual rate and was divided by 12 to convert to a monthly rate since the Microsoft log returns are monthly. The excess Microsoft returns are the log-returns minus the T-bill rates. The variable X:MS_1 in the MINITAB worksheet contains the excess Microsoft returns calculated in this way.

The S&P 500 index was used as the market price and treated the same way as the Ford prices. That is, the S&P 500 prices were logged and differed to get market returns and then the T-bill rates, divided by 12, were subtracted to get excess market returns. The variable X:S&P_1 in the MINITAB worksheet contains the excess market returns.

[5]Interest rates are return rates. Thus, we use the T-bill rates themselves as the risk-free returns. One does *not* take logs and difference the T-bill rates as if they were prices.

After computing the excess market and Microsoft returns as just described, the excess Microsoft returns were regressed on the excess market returns using the "Fitted line" command in MINITAB. Of course, the "Regression" command could have been used itself since "Fitted line" and "Regression" both can be used for regression with one predictor. The fitted line plot is Figure 7.4. The fitted line output and the output from doing the analysis with the "Regression" command are also given below.

For Microsoft, we find that

$$\hat{\beta} = 1.44$$

and

$$\hat{\alpha} = .012.$$

Since the standard error of X:MS_1 (i.e., of $\hat{\beta}_j$) is 0.317, a 95% confidence interval for β_j is $1.44 \pm (2)(.317)$ or $(.81, 2.07)$.[6] The p-value for X:MS_1 is 0.000. This p-value is for testing the null hypothesis that $\beta_j = 0$, so it is not surprising that the null hypothesis is strongly rejected. We do not expect the beta of a stock to be zero, so the outcome of this test should come as no surprise.

The test that $\alpha = 0$ has a p-value of 0.441 so we can accept the null hypothesis. This implies that the data are consistent with the CAPM. Moreover, $\hat{\sigma}_\epsilon^2 = 0.01381$, the mean square residual error.

Here is the MINITAB plot.

Here is the MINITAB regression output.

Model with intercept:

```
Regression Analysis: X:MS_1 versus X:S&P_1


The regression equation is
X:MS_1 = 0.0119 + 1.44 X:S&P_1

Predictor        Coef     SE Coef          T        P
Constant      0.01190     0.01532       0.78    0.441
X:S&P_1        1.4411      0.3174       4.54    0.000

S = 0.1175      R-Sq = 26.2%     R-Sq(adj) = 24.9%
```

[6] Here "2" is used as an approximate t-value. The exact t-value is 2.0017. This value can be found in MINITAB. Go to the calc menu, then probability distrbutions, then "t." Use "inverse cumulative probability" with "noncentrality parameter" equal to 0 and "input constant" equal to .975 (for a 95% confidence interval).

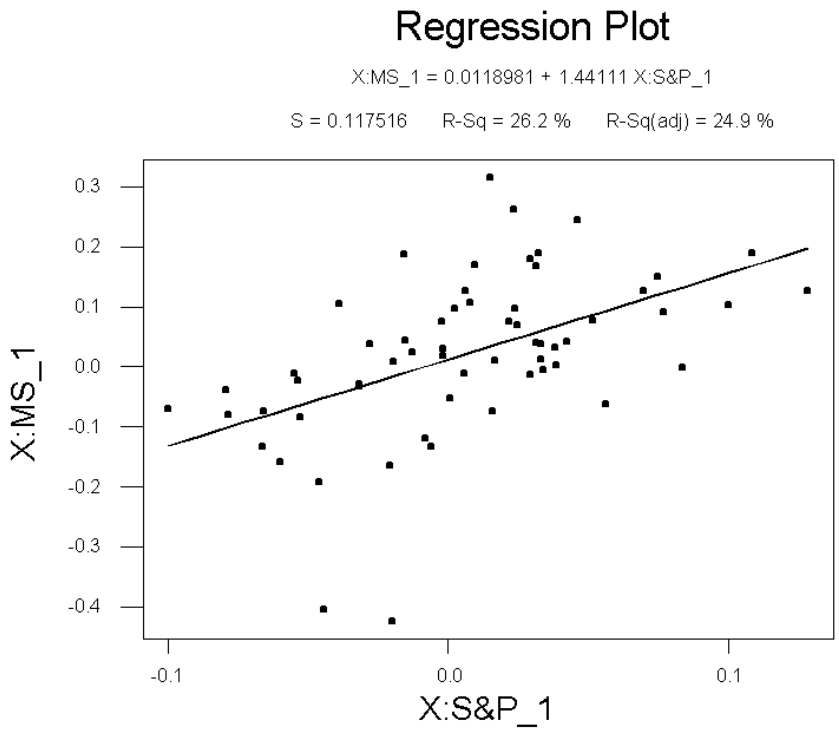


Figure 7.4: *Least squares line fit to the Microsoft data.*

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1     0.28464     0.28464     20.61    0.000
Residual Error    58     0.80098     0.01381
Total             59     1.08562

Unusual Observations
Obs    X:S&P_1     X:MS_1         Fit      SE Fit    Residual    St Resid
 49      0.129     0.1258      0.1973      0.0416     -0.0715       -0.65 X
 50     -0.044    -0.4040     -0.0518      0.0222     -0.3521       -3.05R
 58     -0.020    -0.4249     -0.0166      0.0173     -0.4083       -3.51R
 59      0.015     0.3164      0.0337      0.0154      0.2828        2.43R

R denotes an observation with a large standardized residual
X denotes an observation whose X value gives it large influence.
```

If we assume that $\alpha = 0$, then we can refit the model using a no intercept

model. This is done with MINITAB's regression program by NOT choosing the "fit intercept" option; the default is to choose this option so you need to go to "options" and unchoose that option. Here is the MINITAB output when fitting a no intercept model.

Notice that the R^2 (R-sq) value for the regression is 26.2%. The interpretation of R^2 is the percent of the variance in the excess returns on Microsoft that is due to excess returns on the market. In other words, 26.2% of the risk is due to systematic or market risk ($\beta_j^2\sigma_M^2$). The remaining 73.8% is due to unique or nonmarket risk (σ_ϵ^2).

Model without intercept:

```
Regression Analysis: X:MS_1 versus X:S&P_1

The regression equation is
X:MS_1 = 1.48 X:S&P_1

Predictor        Coef     SE Coef          T        P
Noconstant
X:S&P_1        1.4755      0.3133       4.71    0.000

S = 0.1171

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1     0.30432     0.30432     22.19    0.000
Residual Error    59     0.80931     0.01372
Total             60     1.11362

Unusual Observations
Obs    X:S&P_1     X:MS_1         Fit      SE Fit    Residual    St Resid
 15      0.100     0.1040      0.1479      0.0314     -0.0439       -0.39 X
 30     -0.100    -0.0705     -0.1473      0.0313      0.0767        0.68 X
 33      0.109     0.1906      0.1606      0.0341      0.0300        0.27 X
 45      0.084    -0.0017      0.1240      0.0263     -0.1257       -1.10 X
 49      0.129     0.1258      0.1899      0.0403     -0.0640       -0.58 X
 50     -0.044    -0.4040     -0.0653      0.0139     -0.3387       -2.91R
 58     -0.020    -0.4249     -0.0292      0.0062     -0.3958       -3.38R
 59      0.015     0.3164      0.0223      0.0047      0.2941        2.51R

R denotes an observation with a large standardized residual
X denotes an observation whose X value gives it large influence.
```

Now $\hat{\beta} = 1.48$ and $\hat{\sigma}_\epsilon^2 = 0.0137$ which are small changes from the value from the intercept model. There is some controversy about how one should define R^2 for a no intercept model.[7] MINITAB basically punts by not reporting an R^2 value for a no intercept model.

[7]The problem is that R^2 as originally defined compares a given model with the simple model that only has an intercept. The controversy is what simple model to use for the comparison when the given model does not have an intercept.

7.6.1 Regression using returns instead of excess returns

Often, as an alternative to regression using excess returns, the returns on the asset are regressed on the returns on the asset.

Figure 7.5 show the least squares line using returns instead of excess returns. The estimate of beta has changed from 1.44 to 1.38. The new value, 1.38, is well within the old confidence interval of (.81, 2.07) showing that there is little difference between using returns and using excess returns.

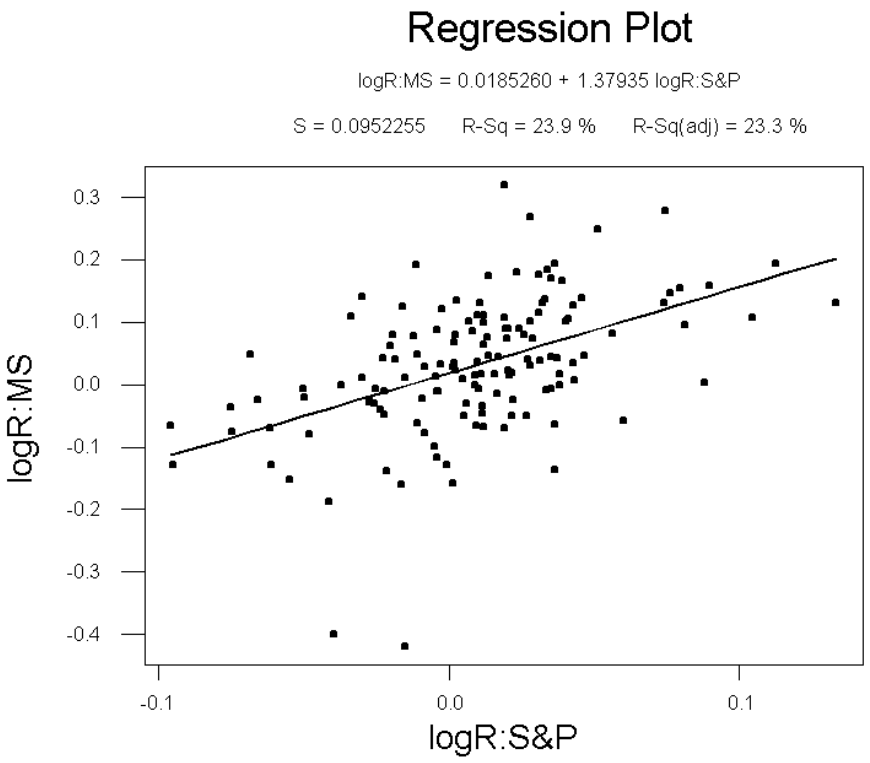


Figure 7.5:

7.6.2 Interpretation of alpha

If α is nonzero then the security is mispriced, at least according to CAPM. If $\alpha > 0$ then the security if underpriced; the returns are too large on average.

This is an indication of an asset worth purchasing. Of course, one must be careful. If we reject the null hypothesis that $\alpha = 0$, all we have done is shown that the security was mispriced *in the past.*

Since for the Microsoft data we accepted the null hypothesis that α is zero, there is no evidence that Microsoft was mispriced.

Warning: If we use returns rather than excess returns as discussed in Section 7.6.1, then intercept of the regression equation does *not* estimate α, so one cannot test whether α is zero by testing the intercept.

7.7 Summary

The CAPM assumes that prices are in equilibrium, that everyone has the same forecasts of returns, and that everyone uses the principles of portfolio selection introduced in Chapter 5.

The CAPM assumptions imply that everyone will hold risk efficient portfolios which mix the tangency portfolio and risk-free assets. This fact implies that the market portfolio will equal the tangency portfolio. A further consequence is that the Sharpe ratio for any efficient portfolio will equal the Sharpe ratio of the market portfolio:

$$\frac{\mu_R - \mu_f}{\sigma_R} = \frac{\mu_M - \mu_f}{\sigma_M}, \tag{7.18}$$

where R is the return on any efficient portfolio. Equation (7.18) can be rearranged to give the CML which is

$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R.$$

The CML applies only to efficient portfolios.

Another consequence of CAPM assumptions is the SML which applies to any security, say the jth, and is

$$\mu_j = \mu_f + (\mu_M - \mu_f)\beta_j.$$

Here β_j is the "independent variable" of this linear relationship and measures the riskiness of the security. μ_j is the "dependent variable."

The security characteristic line is a model for how actual returns are generated. (The SML only described expected returns.) The security characteristic line is

$$R_{jt} = \mu_f + \beta_j(R_{Mt} - \mu_f) + \epsilon_{jt}.$$

The variance of ϵ_{jt} is $\sigma^2_{\epsilon j}$. The security characteristic line implies that the risk of the jth asset can be decomposed into market and non market risks:

$$\sigma_j = \sqrt{\beta_j^2\sigma_M^2 + \sigma^2_{\epsilon j}}.$$

If one assumes that ϵ_{jt} is uncorrelated with $\epsilon_{j't}$ for $j \neq j'$ (that is, for two different securities), then non market risk can be eliminated by portfolio diversification.

Since the security characteristic line is a regression model it can be used to estimate β_j and $\sigma^2_{\epsilon j}$. The R^2 value of the regression estimates the proportion of the total risk (σ_j^2) due to market risk, i.e., it estimates $\beta_j^2\sigma_M^2/\sigma_j^2$. Also, $1 - R^2 = \sigma^2_{\epsilon j}/\sigma_j^2$ is the proportion of total risk that is non-market and can be removed by diversification.

7.8 References

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Sharpe, W. F., (1964), Capital asset prices: a theory of market equilibrium under conditions of risk, *J. of Finance*, 19, 425–442.

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Chapter 8

Pricing Options: 3/11/02

8.1 Introduction

The European call options mentioned in Chapter 1 are one example of the many derivatives now on the market. A *derivative* is a financial instrument whose value is derived from the value of some underlying instrument such an interest rate, foreign exchange rate, or stock price.

A call option gives one the right to buy a certain asset such as a stock at the *exercise* or *strike* price, while a *put* option gives one the right to sell the asset at the exercise price. An option has an *exercise date*, which is also called the *strike date, maturity,* or *expiration date*. American options can be exercised at any time up to their exercise date, but European options can be exercised only at their exercise date. European options are easier to analyze than American options since one does not need to consider the possibility of early exercise.

Many complex types of derivatives cannot be priced by a simple formula such as the Black-Scholes formula. Rather, these options must be priced numerically, for example, using computer simulation. Numerical pricing is an advanced topic that will not be covered in this course.

In this chapter we will discuss the main ideas behind the pricing of options. I will not actually prove the Black-Scholes formula, since that derivation requires advanced mathematics. However, I will present a heuristic derivation of that formula to give you an intuitive understanding of option pricing. For lack of time, we will only study European call options in detail. However, Black-Scholes type formulas exist for other derivatives as well. We will give one example of a put option. The book *Introduction to Futures and Options Markets* by Hull is a nice overview of the options markets

and the uses of options. *Financial Calculus* by Baxter and Rennie discuss the mathematics of many types of derivatives.

Why do companies purchase options and other derivatives? The answer is simple: to manage risk. In its 2000 Annual Report, the Coca Cola Company wrotes

> Our company uses derivative financial instruments primarily to reduce our exposure to adverse fluctuations in interest rates and foreign exchange rates and, to a lesser extent, adverse fluctuations in commodity prices and other market risks. We do not enter into derivative finanicial instruments for trading pruposes. As a matter of policy, all our derivative positions are used to reduce risk by hedging an underlying economic exposure. Because of the high correlation between the hedging instrument and the underlying exposure, fluctuations in the value of the instruments are generally offset by reciprocal changes in the value of the underlying exposure. The derivatives we use are straightforward instruments with liquid markets.

Derivatives can and have been used to speculate, but that is not their primary purpose. The intent of this quote is clear. The Company was assuring its stockholders that it is using derivatives to manage risk, not to gamble.

8.2 Call options

Suppose that you have purchased a European call option on 100 shares of Stock A with a exercise price of \$70. At the expiration date, suppose that Stock A is selling at \$73. The option allows you to purchase the 100 shares for \$70 and to immediately sell them for \$73, with a gain of \$300 on the 100 shares. Of course, the net profit for purchasing the option isn't \$300 since you had to pay for the option. If the option cost \$2/share, then you paid \$200 for the option. Moreover, you paid the \$200 up front but only got the \$300 at the expiration date. Suppose that the expiration date was 3 months after the purchase date and the continuously compounded risk-free rate is 6% per annum or 1.5% for 3 months. Then the dollar value of your net profit is

$$\exp(-.015)300 - 200 = 95.53$$

at the time of purchase and is

$$300 - \exp(.015)200 = 96.98$$

at the exercise date.

We will use the notation $(x)_+ = x$ if $x > 0$ and $= 0$ if $x \leq 0$. With this notation, the value of a call at exercise date is

$$(S_T - E)_+,$$

where E is the exercise date and S_T is stock's price on the exercise date, T.

A call is never exercised if the strike price is greater than the price of the stock, since exercising the option would amount to buying the stock for more than it would cost on the market. If a call is not exercised, then one loses the cost of buying the option.

One can lose money on an option even if it is exercised, because the amount gained by exercising the option might be less than the cost of buying the option. In the example above, if Stock A were selling for \$71 at the exercise data, then one would exercise the option and gain \$100. This would be less than the \$200 paid for the option. Even though exercising the option results in a loss, then loss is less than it would be if the option were not exercised. An option should always be exercised if $(S_t - E)$ is positive.

8.3 The law of one price

The "law of one price" states that if two financial instruments have exactly the same payoffs, then they will have the same price. This principle is used to price options. To valuate an option, one must find a portfolio or a self-financing[1] trading strategy with a *known price* and which has exactly the same payoffs as the option. The price of the option is then known; it must be the same as the price of the portfolio or self-financing trading strategy.

Here's a simple example of pricing by the law of one price. Suppose stock in company A sells at \$100/share. The risk-free rate of borrowing is 6% compounded annually. Consider a *futures contract* obliging one party to sell to the other party one share of Company A exactly one year from now at a price P. (No money changes hands now.) What is the fair market price, i.e., what should P be?

Note that this contract is not an option because the sale *must* take place. Therefore, this contract is easier to price than an option and is a good example to use here to illustrate the law of one price. It would seem that P should depend on the expected price of company A stock one year from

[1] A trading strategy is "self-financing" if it requires no investment other than the initial investment. After the initial investment, any further purchases of assets are financed by the sale of other assets or by borrowing.

now. Surprisingly, this is not the case. Consider the following strategy. The party that, one year from now, must sell the share of company A can borrow \$100 and buy one share now; this involves no capital since the share is purchased with borrowed money. A year from now that party sells the share for P dollars and pays back \$106 (principle plus interest) to the lender, who could be a third party. The profit is $P - 106$. The fair profit is 0 since no capital was used and there is no risk. Therefore, P should be \$106.

Consider what would happen if P were not \$106. You should be able to see that any other value of P besides \$106 would lead to unlimited risk-free profits. As investors rushed in to take advantage of this situation, the market would immediately correct the value of P to be \$106.

8.3.1 Arbitrage

Arbitrage is the making of a guaranteed risk-free profit by trading in the market with no invested capital[2]. In other words, arbitrage is a "free lunch." The *arbitrage price* of a security is the price that guarantees no arbitrage opportunities. The law of one price is equivalent to stating that the market is free of arbitrage opportunities, that is, that there are no free lunches. Arbitrage pricing is the same as pricing by the law of one price. The price of \$106 that we just derived in the example of the future contract is, therefore, the arbitrage price.

8.4 Time value of money and present value

"Time is money" is an old adage that is still true. A dollar a year from now is worth less to us than a dollar now. In finance it is essential that we be able to convert value in future payments to their present values, or vice versa. For example, we saw in Section 8.3 that the arbitrage enforced future price of a stock is simply the present price converted into a "future value" by multiplying by $1 + r$.

Let r be the risk-free annual interest rate. Then the "present value" of \$D dollars one year from now is \$D$/(1 + r)$ without compounding or \$D$\exp(-r)$ under continuous compounding. Another way of stating this is that \$D dollars now is worth \$(1+r)D dollars a year from now without compounding, or \${$\exp(r)$} D dollars a year from now under continuous

[2]Investing in risk-free T-bills guarantees a positive net return but is *not* arbitrage since capital is invested.

compounding. When $D is a future cash flow, then its present value is also called a *discounted* value and r is the discount rate.

The distinction between simple and compounding is not essential since an interest rate of r without compounding is equivalent to an interest rate of r' with continuously compounding where

$$1 + r = \exp(r')$$

so that

$$r = \exp(r') - 1 \text{ or } r' = \log(1 + r).$$

We will work with both simple and compound interest, whichever is most convenient.

Examples

If $r = 5\%$, then $r' = \log(1.05) = .0488$ or 4.88%. If $r' = 4\%$, then $r = \exp(.04) - 1 = 1.0408 - 1$ or 4.08%. In general, $r > r'$

Occasionally, we will simplify life by making the unrealistic assumption that $r = 0$ so that present and future values are equal. This simplifying assumption allows us to focus on other concepts besides discounting.

8.5 Pricing calls — a simple binomial example

We will start our study of options with a very simple example. Suppose that a stock is currently selling for $80. At the end of one time period it can either have increased to $100 or decreased to $60. What is the current value of a call option that allows one to purchase one share of the stock for $80, the exercise price, after one time period?

At the end of the time period, the call option will be worth $20 ($100 − $80) if the stock has gone up and worth $0 dollars if the stock has gone down. See Figure 8.1. However, the question is "what is the option worth *now*?" This question if vital since the answer is, of course, the fair market price for the option at the current time.

One might think that the current value of the option depends on the probability that the stock will go up. However, this is not true. The current value of the option depends only on the rate of risk-free borrowing. For simplicity, we will assume that this rate is 0; later we will see how to valuate

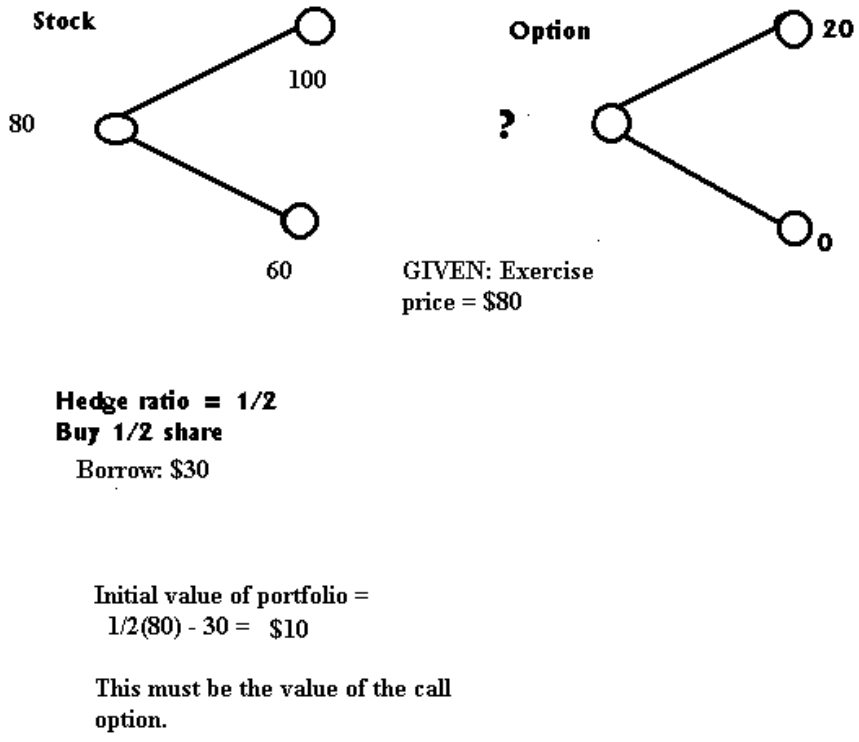

Figure 8.1: *Example of one-step binomial option pricing. For simplicity, it is assumed that the interest rate is 0. The portfolio of 1/2 share of stock and −$30 of risk-free assets replicates the call option.*

options when the rate of interest is positive. It turns out that the value of the option is \$10. How did I get this value?

Consider the following investment strategy. Borrow \$30 and buy one-half of a share of stock. The cost upfront is \$40 − \$30 = \$10, so the value now of the portfolio is \$10. If after one time period the stock goes up, then the portfolio is worth $100/2 - 30 = 20$ dollars. If the stock goes down, then the portfolio is worth $60/2 - 30 = 0$ dollars. Thus after one time period, the portfolio's value will be *exactly* the same as the value of the call option, no matter which way the stock moves. By the law of one price, the value of the call option *now* must be the same as the value *now* of the portfolio, which is \$10.

Let's summarize what we have done. We have found a portfolio of the stock and the risk-free asset that *replicates* the call option. The current value of the portfolio is easy to calculate. Since the portfolio replicates the option, the option must have the same value as the portfolio.

Suppose we have just sold a call option. By purchasing this portfolio we have hedged the option. By hedging is meant that we have eliminated all risk, because the net return of selling the option and purchasing the portfolio is exactly 0 no matter what happens to the stock price.

How did I know that the portfolio should be 1/2 share of stock and −\$30 in cash? I didn't use trial-and-error; that would have been tedious. Rather, I used the following logic. First, the volatility of one share of the stock is \$100 − \$60 = \$40 while the volatility of the option is \$20 − \$0 = \$20. The ratio of the volatility of the option to the volatility of the stock is 1/2; this is called the hedge ratio. If the portfolio is to exactly replicate the option, then the portfolio must have exactly the same volatility as the option; this means the portfolio must have one-half a share.

Key point: The number of shares in the portfolio must equal the hedge ratio, where

$$\text{hedge ratio} = \frac{\text{volatility of option}}{\text{volatility of stock}}.$$

So now we know why the portfolio holds 1/2 share of stock? How was it determined that \$30 should be borrowed? If the stock goes down, the portfolio is worth \$30 minus the amount borrowed. But we want the portfolio's value to equal that of the option, which is \$0. Thus, the amount borrowed is \$30.

Key point: We can determine the amount borrowed by equating the value of the portfolio when the stock goes down to the value of the option when the stock goes down. (Alternatively, we could equate the value of

the portfolio to the value of the option when the stock goes up. This would tell us that \$50 minus the amount borrowed equals \$20, or that \$30 must be borrowed.)

Now suppose that the interest rate is 10%. Then, we borrow \$30/(1.1) = \$27.27 so that the amount owed after one year is \$30. The cost of the portfolio is $40 - 30/1.1 = \$12.7273$. Thus, the value of the option is \$12.7273 if the risk-free rate of interest is 10%. This value is higher than the value of the option when the risk-free rate is 0, because the initial borrowing used by the self-financing strategy is more expensive when the interest rate is higher.

Here's how to valuate one-step binomial options for other values of the parameters. Suppose the current price is s_1 and after one time period the stock either goes up to s_3 or down to s_2. The exercise price is E. The risk-free rate of interest is r. It is assumed that $s_2 < E < s_3$, so the option is exercised if and only if the stock goes up.[3] Therefore, the hedge ratio is

$$\delta = \frac{s_3 - E}{s_3 - s_2}. \tag{8.1}$$

This is the number of shares of stock that are purchased; the cost is $\delta\, s_1$. The amount borrowed is

$$\frac{\delta s_2}{1+r}, \tag{8.2}$$

and the amount that will be paid back to the lender will be δs_2. Therefore, the price of the option is

$$\delta \left\{ s_1 - \frac{s_2}{1+r} \right\} = \frac{s_3 - E}{s_3 - s_2} \left\{ s_1 - \frac{s_2}{1+r} \right\}. \tag{8.3}$$

If the stock goes up, then the option is worth $(s_3 - E)$ and the portfolio is also worth $(s_3 - E)$. If the stock goes down, both the option and the portfolio are worth 0. Thus, the portfolio does replicate the option.

Example

In the example analyzed before, $s_1 = 80$, $s_3 = 100$, $s_2 = 60$, and $E = 80$. Therefore,

$$\delta = \frac{100 - 80}{100 - 60} = \frac{1}{2}.$$

[3] This assumption is very reasonable for the following reasons. If $s_2 < s_3 \leq E$, then the option will never be exercised under any circumstances. We are certain the option will be worthless, so its price must be 0. If $E \leq s_2 < s_3$, then the option will always be exercised; it really isn't an option, it is a futures contract. We have already seen how to valuate a futures contract—that was done in Section 8.3.

The price of the option is

$$\frac{1}{2}\left\{80 - \frac{60}{1+r}\right\},$$

which is \$10 if $r = 0$ and \$12.7273 if $r = 0.1$. The amount borrowed is

$$\frac{\delta s_2}{1+r} = \frac{(1/2)60}{1+r} = \frac{30}{1+r},$$

which is \$30 if $r = 0$ and \$27.27 if $r = .1$.

8.6 Two-step binomial option pricing

A one-step binomial model for a stock price may be reasonable for very short maturities. For longer maturities, multiple-step binomial models are needed. A multiple-step model can be analyzed by analyzing the individual steps, going backwards in time.

To illustrate multi-step binomial pricing, consider the two-step model of a European call option in Figure 8.2. The option matures after the second step. The stock price can either go up \$10 or down \$10 on each step. Right now we will assume that $r = 0$ for simplicity.

Using the pricing principles just developed and working backwards, we can fill in the question marks in Figure 8.2. See Figure 8.3. For example, at node B, the hedge ratio is $\delta = 1$ so we need to own one share which at this node is worth \$90. Also, we need to have borrowed $\delta s_2/(1+r) = (1)(80)/(1+0) =$ \$80 so that our portfolio has the same value at nodes E and F as the option, that is, the portfolio should be worth \$0 at node E and \$20 at node F. Since at node B we have stock worth \$90 and the risk-free asset worth $-$\$80, the net value of our portfolio is \$10. By the same reasoning, at node C the hedge ratio is 0 and we should have no stock and no borrowing, so our portfolio is worth \$0.

We can see in Figure 8.3 that at the end of the first step the option is worth \$10 is the stock is up (node B) and \$0 if it is down (node C). Applying one-step pricing at node A, at the beginning of trading the hedge ratio is $1/2$ and we should own $1/2$ share of stock (worth \$40) and we should have borrowed \$35. Therefore, the portfolio is worth \$5 at node A, which proves that \$5 is the correct price of the option.

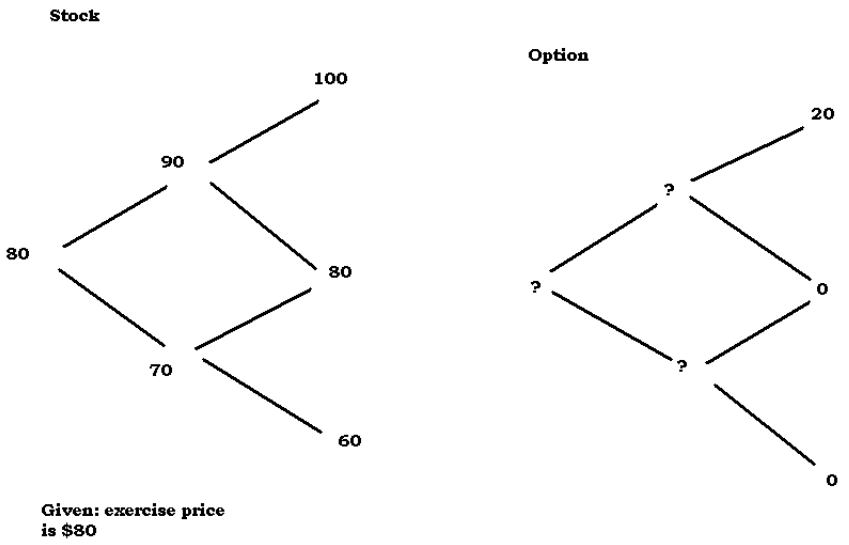

Figure 8.2: *Two-step binomial model for option pricing.*

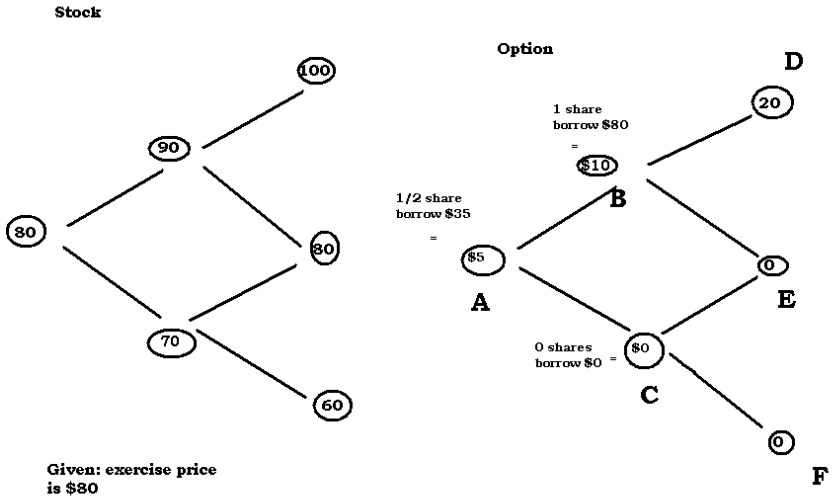

Figure 8.3: *Pricing the option by backwards induction.*

Note the need to work backwards. We could not apply one-step pricing at node A until we had already found the value of the portfolio (and of the option) at nodes B and C.

Let's show that our trading strategy is self-financing. To do this we need to show that we invest no money other than the initial \$5. Suppose that the stock is up on the first step, so we are at node B. Then our portfolio is worth $\$90/2 - \$35$ or \$10. At this point we borrow \$45 and buy another half-share for \$45; this is self-financing. If the stock is down on the first step, we self the half share of stock for \$35 and buy off our debt; again the step is self-financing.

8.7 Arbitrage pricing by expectation

It was stated earlier that one prices an option by arbitrage, that is, the price is determined by the requirement that the market be arbitrage-free. The expected value of the option is *not* used to price the option. In fact, we do not even consider the probabilities that the stock moves up or down.

However, there is a remarkable result showing that arbitrage pricing *can* be done using expectations. More specifically, there exists probabilities of the stock moving up and down such that the arbitrage price of the option *is* equal to the expected value of the option according to these probabilities. Whether these are the "true" probabilities of the stock moving up or down is irrelevant. The fact is that these probabilities give the correct arbitrage price when they are used to calculate expectations.

Let "now" be time 0 and let "one-step ahead" be time 1. Because of the time value of money, the present value of \$D dollars at time 1 is $\$D/(1+r)$ where r is the interest rate. Let $f(2) = 0$ and $f(3) = s_3 - E$ be the values of the option if the stock moves up or down, respectively. We will now show that there is a value of q between 0 and 1, such that the present value of the option is

$$\frac{1}{1+r}\{qf(3) + (1-q)f(2)\}. \tag{8.4}$$

The quantity in (8.4) is the present value of the expectation of the option at time 1. To appreciate this, notice that the quantity in curly brackets is the value of the option if the stock goes up times q, which is the arbitrage determined "probability" that the stock goes up, plus the option's value if the stock goes down times $(1-q)$. Thus the quantity in curly brackets is the

expectation of the value's option at the end of the holding period. Dividing by $1+r$ converts this to a "present value."

Okay, how do we find this magical value of q? That's easy. We know that q must satisfy

$$\frac{1}{1+r}\{qf(3)+(1-q)f(2)\} = \frac{s_3-E}{s_3-s_2}\left\{s_1 - \frac{s_2}{1+r}\right\}. \tag{8.5}$$

since the left hand side of this equation is (8.4) and the right-hand side is the value of the option according to (8.3). Substituting $f(2)=0$ and $f(3)=s_3-E$ into (8.5) we get an equation that can be solved for q to find that

$$q = \frac{(1+r)s_1 - s_2}{s_3 - s_2}. \tag{8.6}$$

We want q to be between 0 and 1 so that it can be interpreted as a probability. From (8.6) one can see that $0 \le q \le 1$ if $s_2 \le (1+r)s_1 \le s_3$. Why should the latter hold?

We will show that $s_2 \le (1+r)s_1 \le s_3$ is required in order for the market to be arbitrage-free. If we invest s_1 in a risk-free asset at time 0, then the value of our holdings at time 1 will be $(1+r)s_1$. If we invest s_1 in the stock, then the value of our holdings at time 1 will be either s_2 or s_3. If $s_2 \le (1+r)s_1 \le s_3$ were not true, then there would be an arbitrage opportunity. For example, if $(1+r)s_1 < s_2 \le s_3$, then could borrow at the risk-free rate and invest the borrowed money in the stock with a guaranteed profit; at time 1 we would pay back $(1+r)s_1$ and receive *at least* s_2 which is greater that $(1+r)s_1$.

Exercise: How would we make a guaranteed profit if $s_2 \le s_3 < (1+r)s_1$?

Answer: Sell the stock short and invest the s_1 dollars in the risk-free asset. At the end of the holding period (maturity) receive $(1+r)s_1$ from the risk-free investment and buy the stock for at most $s_3 < (1+r)s_1$.

Thus, the requirement that the market be arbitrage-free ensures that $0 \le q \le 1$.

8.8 A general binomial tree model

The material in this section follows Chapter 2 of *Financial Calculus* by Baxter and Rennie. Consider a possibility non-recombinant[4] tree as seen in Figure 8.4

[4]The tree would be *recombinant* if the stock prices at nodes 5 and 6 were equal so that these two nodes could be combined.

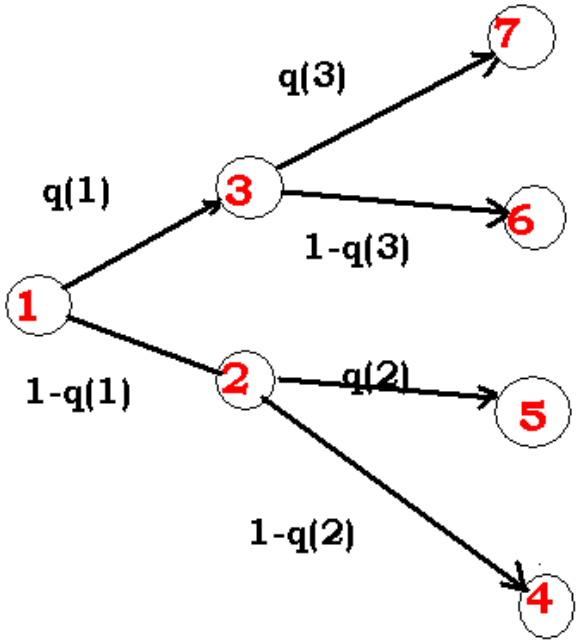

Figure 8.4: *Two-step non-recombinant tree. The $q(j)$ is the risk-neutral probability at node j of the stock moving upward.*

Assume that:

- At the jth node the stock is worth s_j and the option is worth $f(j)$.
- The jth node leads to either the $2j+1$th node or the $2j$th node after one time "tick."
- The actual time between ticks is δt.
- Interest is compounded continuously at a fixed rate r so that B_0 dollars now is worth $\exp(rn\,\delta t)B_0$ dollars after n time ticks. (Or, B_0 dollars after n ticks is worth $\exp(-rn\delta t)B_0$ dollars now.)

Then at node j:

- The value of the option is

$$f(j) = \exp(-r\,\delta t)\Big\{q_j f(2j+1) + (1-q_j) f(2j)\Big\}.$$

  where

- The arbitrage determined q_j is

$$q_j = \frac{e^{r\,\delta t} s_j - s_{2j}}{s_{2j+1} - s_{2j}}. \tag{8.7}$$

- The number of shares of stock to be holding is

$$\phi_j = \frac{f(2j+1) - f(2j)}{s_{2j+1} - s_{2j}} = \text{hedge ratio}.$$

- Denote the amount of capital to hold in the risk-free asset by ψ_j; typically ψ_j is negative because money has been borrowed. Since the portfolio replicates the option, at node j the option's value, which is $f(j)$, must equal the portfolio's value which is $s_j\phi_j + \psi_j$. Therefore,

$$\psi_j = \{f(j) - \phi_j s_j\}. \tag{8.8}$$

  (ψ_j changes in value to $e^{r\,\delta t}\{f(j) - \phi_j s_j\}$ after one more time tick).

Expectations for paths along the tree are computed using the q_j's. The probability of any path is just the product of all the probabilities along the path.

An example

The tree for the example of Section 8.6 is shown in Figure 8.5. Because $r = 0$ is assumed and because the stock moves either up or down the same amount ($10), the q_j are all equal to $1/2$.[5]

The probability of each full path from node 1 to one of nodes 4, 5, 6, or 7 is $1/4$.

[5]It follows from (8.7) that whenever $r = 0$ and the up moves, which equal $s_{2j+1} - s_j$, and down moves, which equal $s_j - s_{2j}$, are of equal length, then $q_j = 1/2$ for all j.

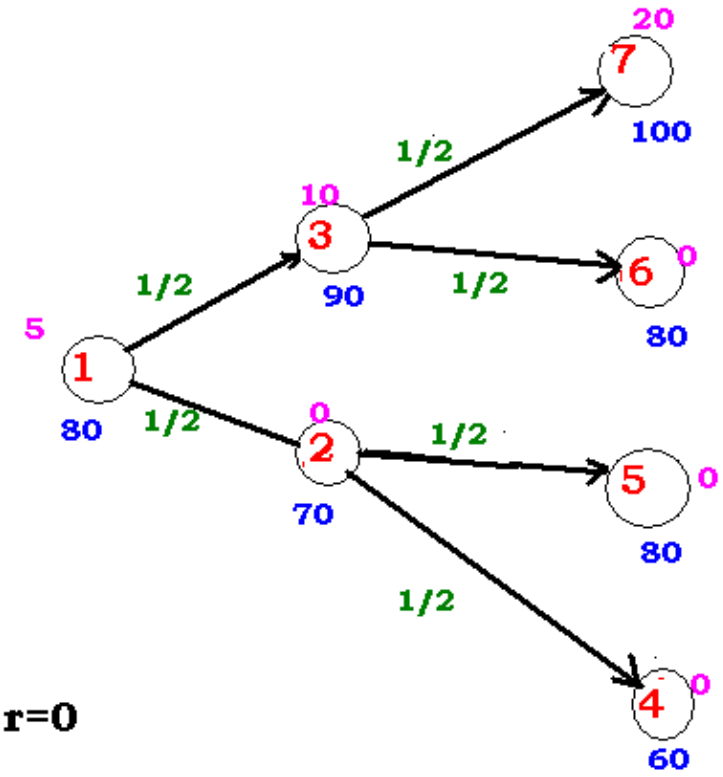

Figure 8.5: *Two-step example with pricing by probabilities. Red is node number. Blue is value of the stock. Magenta is value of the option. Path probabilites are in dark green. The exercise price is $80.*

Given the values of the option at nodes 4, 5, 6, and 7, it is easy to compute the expectations of the option's value at other nodes. These expectations are shown in magenta in Figure 8.5.

The path probabilities are independent of the exercise price, since they depend only on the prices of stock at the nodes and on r. Therefore, it is easy to price options with other exercise prices.

Exercise

Assuming the same stock price process as in Figure 8.5, price the call option with an exercise price of \$70.

Answer: Given this exercise price, it is clear that the option is worth \$0, \$10, \$10, and \$30 dollars at nodes 4, 5, 6, and 7, respectively. Then we can use expectation to find that the option is worth \$5 and \$20 at nodes 2 and 3, respectively. Therefore, the option's value at node 1 is \$12.50; this is the price of the option.

8.9 Martingales

A *martingale* is a probability model for a fair game, that is, a game where the expected changes in one's fortune are always zero. More formally, a stochastic process $Y_0, Y_1, Y_2, \ldots$ is a martingale if

$$E(Y_{t+1}|Y_t) = Y_t$$

for all t.

Example

Suppose that on each toss of a fair coin we wager half of our fortune that a head will appear. If our fortune at time t is Y_t, then we win or loss $Y_t/2$ with probability $1/2$. Thus, our fortune at time $t+1$ is either $Y_t/2$ or $(3/2)Y_t$, each with probability $1/2$. Therefore, $E\{Y_{t+1}|Y_t\} = (1/2)(Y_t/2)+(1/2)(3/2)Y_t = Y_t$ so that our sequence of fortunes is a martingale.

Let P_t, $t = 0, 1, \ldots$ be the price of the stock at the end of the tth step in a binomial model. Then $P_t^* := \exp(-rt\,\delta t)P_t$ is the discounted price process.

Key fact: Under the $\{q_j\}$ probabilities, the discounted price process P_t^* is a martingale.

To see that P_t^* is a martingale, we calculate using the definition (8.7) of q_j:

$$\begin{aligned} E(P_{t+1}|P_t = s_j) &= q_j s_{2j+1} + (1-q_j)s_{2j} \\ &= s_{2j} + q_j(s_{2j+1} - s_{2j}) \\ &= s_{2j} + \{\exp(r\,\delta t)s_j - s_{2j}\} = \exp(r\,\delta t)s_j. \end{aligned}$$

This holds for all values of s_j. Therefore,

$$E(P_{t+1}|P_t) = \exp(r\,\delta t)P_t,$$

so that

$$E\{\exp(-r(t+1)\,\delta t)P_{t+1}|P_t)\} = \exp(-rt\,\delta t)P_t,$$

or

$$E(P_{t+1}^*|P_t^*) = P_t^*.$$

This shows that P_t^* is a martingale.

8.9.1 Martingale or risk-neutral measure

Any set of path probabilities, $\{p_j\}$, is called a *measure* of the process. The measure $\{q_j\}$ is called the *martingale measure* or the *risk-neutral measure*. We will also call $\{q_j\}$ the risk-neutral path probabilities.

8.9.2 The risk-neutral world

If all investors were risk-neutral, that is, indifferent to risk, then there would be no risk premiums and all expected asset prices would rise at the risk-free rate. Therefore, all discounted asset prices, with discounting at the risk-free rate, would be martingales.

We know that we do not live in such a risk-free world, but there is a general prinicple that expectations taken with respect to a risk-neutral model give correct, i.e., arbitrage-free, prices of options and other financial instruments.

Example

In Section 8.3 it was argued that if a stock is selling at \$100/share and the risk-free interest rate is 6%. then the correct future delivery price of a share one year from now is \$106. We can now calculate this value using the risk-neutral measure—in the risk-neutral world, the expected stock price will increase to exactly \$106 one year from now.

8.10 Trees to random walks to Brownian motion

8.10.1 Getting more realistic

Binomial trees are useful because they illustrate several important concepts, in particular:

- arbitrage pricing
- self-financing trading strategies
- hedging
- computation of arbitrage prices by expectations with respect to an appropriate set of probabilities called the risk-neutral measure

However, binomial trees are not realistic, because stock prices are continuous, or at least approximately continuous. This lack of realism can be alleviated by increasing the number of steps. In fact, one can increase the number of steps without limit to derive the Black-Scholes model and formula. That is the goal of Section 8.11. The present section will get us closer to that goal.

8.10.2 A three-step binomial tree

Figure 8.6 is a three-step tree where at each step the stock price either goes up \$10 or down \$10. Assume that the risk-free rate is $r = 0$.

Now consider the price of the stock, call it P_t at time t where $t = 0, 1, 2, 3$. Using the risk-neutral path probabilities, which are each $1/2$ in this example, P_t is a *stochastic process*, that is a process that evolves randomly in time. In fact, since P_{t+1} equals $P_t \pm$ \$10, this process is a random walk. We have

$$P_t = P_0 + (\$10)\{2(W_1 + \cdots + W_t) - t\} \tag{8.9}$$

where $W_1, \cdots, W_3$ are independent and W_t equal 0 or 1, each with probability $1/2$. If W_t is 1, then $2W_t - 1 = 1$ and the price jumps up \$10 on the tth step. If W_t is 0, then $2W_t - 1 = -1$the price jumps down \$10.

The random sum $W_1 + \cdots + W_t$ is Binomial$(t, 1/2)$ distributed and so has a mean of $t/2$ and variance equal to $t/4$.

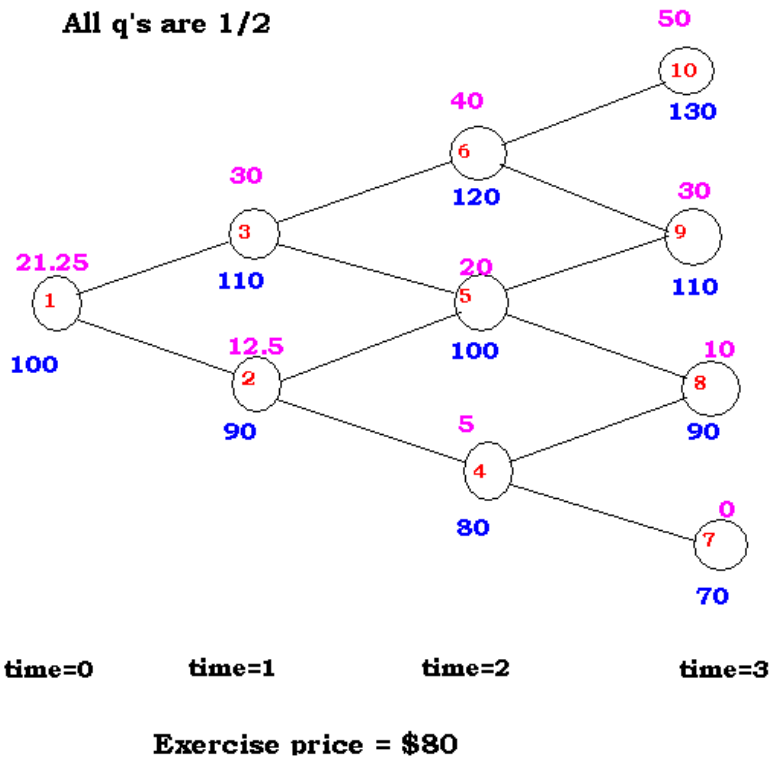

Figure 8.6: *Three-step example of pricing a European call option by probabilities. Red is node number. Blue is value of the stock. Magenta is value of the option. Risk-neutral path probabilites are not shown, but they are all equal to 1/2. The exercise price is $80. The risk-free rate is* $r = 0$.

The value of the call option is

$$E\{(P_3 - E)_+\} \tag{8.10}$$

where x_+ equals x if $x \geq 0$ and equals 0 otherwise. The expectation in (8.10) is with respect to the risk-neutral probabilities. Since $W_1 + W_2 + W_3$ is Binomial$(3, 1/2)$, it equals $0, 1, 2,$ or 3 with probabilities 1/8, 3/8, 3/8, and 1/8, respectively. Therefore,

$$\begin{aligned} E\{(P_3 - E)_+\} &= \frac{1}{8}\Big[\{P_0 - 30 - E + (20)(0)\}_+ + 3\{P_0 - 30 - E + (20)(1)\}_+ \\ &+ 3\{P_0 - 30 - E + (20)(2)\}_+ + \{P_0 - 30 - E + (20)(3)\}_+\Big]. \end{aligned}$$

Examples

If $P_0 = 100$ and $E = 80$, then $P_0 - 30 - E = -10$ and

$$\begin{aligned} E\{(P_3 - E)_+\} &= \frac{1}{8}\Big\{(-10 + 0)_+ + 3(-10 + 20)_+ \\ &+ 3(-10 + 40)_+ + (-10 + 60)_+\Big\} \\ &= \frac{1}{8}(0 + 30 + 90 + 50) = \frac{170}{8} = 21.25 \end{aligned}$$

as seen in Figure 8.6.

Similarly, if $P_0 = 100$ and $E = 100$, then $P_0 - 30 - E = -30$ and

$$\begin{aligned} E\{(P_3 - E)_+\} &= \frac{1}{8}\Big\{(-30 + 0)_+ + 3(-30 + 20)_+ \\ &+ 3(-30 + 40)_+ + (-30 + 60)_+\Big\} \\ &= \frac{1}{8}(0 + 0 + 30 + 30) = \frac{60}{8} = 7.5 \end{aligned}$$

8.10.3 More time steps

Let's consider a call option with maturity data equal to 1. Take the time interval [0, 1] and divide it into n steps, each of length $1/n$. Suppose that the stock price goes up or down $\sigma/\sqrt{n}$ at each step. The step size $\sigma/\sqrt{n}$ being used here was carefully chosen as you will soon see. Then the price after m steps ($0 \leq m \leq n$) when $t = m/n$ is

$$P_{m/n} = P_0 + \frac{\sigma}{\sqrt{n}}\{2(W_1 + \cdots + W_m) - m\}. \tag{8.11}$$

Since, $W_1 + \cdots + W_m$ is Binomial$(m, 1/2)$ distributed, it follows that

$$E(P_t|P_0) = P_0. \tag{8.12}$$

and

$$\text{Var}(P_t|P_0) = \frac{4\sigma^2}{n}\frac{m}{4} = \frac{m}{n}\sigma^2 = t\sigma^2, \tag{8.13}$$

and, in particular,

$$\text{Var}(P_1|P_0) = \sigma^2. \tag{8.14}$$

Moreover, by the central limit theorem, as $n \to \infty$, P_1 converges to a $N(P_0, \sigma^2)$ random variable.

The step size $\sigma/\sqrt{n}$ was chosen so that $\text{Var}(P_1|P_0) = \sigma^2$. Here σ is just a volatility parameter that would, in practice, be estimated from market data. The factor $\sqrt{n}$ in the denominator was "reverse engineered" to make $\text{Var}(P_t|P_0)$ converge to $t\sigma^2$ as $n \to \infty$.

Let E be the exercise price. Remember that the value of an option is the expectation with respect to the risk-neutral measure of the present value of the option at expiration. Therefore, in the limit, as the number of steps goes to ∞, the price of the option converges to

$$E\{(P_0 + \sigma Z - E)_+\} \tag{8.15}$$

where Z is $N(0, 1)$ so that $P_1 = P_0 + \sigma Z$ is $N(P_0, \sigma^2)$.

For a fixed value of n, P_t is a discrete time stochastic process since $t = 0, 1/n, 2/n, \ldots, (n-1)/n, 1$. In fact, as we saw before, for any finite value of n, P_t is a random walk. However, in the limit as the number of steps $n \to \infty$, P_t becomes a continuous time stochastic process. This limit process is called *Brownian motion*. In other words, the continuous time limit of a random walk is Brownian motion.

8.10.4 Properties of Brownian motion

We have seen that Brownian motion is a continuous-time stochastic process that is the limit of discrete-time random walk processes. A Brownian motion process, B_t, starting at 0, i.e., with $B_0 = 0$, has the following mathematical properties:

1. $E(B_t) = 0$ for all t.

2. $\text{Var}(B_t) = t\sigma^2$ for all t. Here σ^2 is the volatility of B_t.

3. Changes over non-overlapping increments are independent. More precisely, if $t_1 < t_2 < t_3 < t_4$ then $B_{t_2} - B_{t_1}$ and $B_{t_4} - B_{t_3}$ are independent.

4. B_t is normally distributed for any t.

If B_0 is not zero, then each of these properties holds for the process $B_t - B_0$, which is the change in B_t from times 0 to t. All of these properties but the last are shared with random walks with mean-zero steps.

8.11 Geometric Brownian motion

Random walks are not realistic models for stock prices, since a random walk can go negative. Therefore, (8.15) is close to but not quite the correct price of the option. To get the correct price we need to make our model more realistic. We saw in Chapter 3 that geometric random walks are much better than random walks as models for stock prices since geometric random walks are always non-negative.

We will now introduce a binomial tree model that is geometric random walk. We do this by making the steps proportional to the current stock price. Thus, if s is the stock price at the current node, then price at the next node is

$$s \exp(\mu/n \pm \sigma/\sqrt{n}) = (s_{\text{up}}, s_{\text{down}}).$$

Notice that the log of the stock price is a random walk since

$$(\log(s_{\text{up}}), \log(s_{\text{down}})) = \log(s) + \frac{\mu}{n} \pm \frac{\sigma}{\sqrt{n}}.$$

Therefore, the stock price process is a geometric random walk. There is a drift if $\mu \neq 0$, but we will see that the amount of drift is irrelevant. We could have set the drift equal to 0 but we didn't to show later that the drift does NOT affect the option's price.

The risk neutral probability of an up jump is by equation (8.7)

$$\begin{aligned} q &= \frac{s\exp(r/n) - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}} \\ &= \frac{\exp(r/n) - \exp(\mu/n - \sigma/\sqrt{n})}{\exp(\mu/n + \sigma/\sqrt{n}) - \exp(\mu/n - \sigma/\sqrt{n})} \\ &\approx \frac{1}{2}\left(1 - \frac{\mu - r + \sigma^2/2}{\sigma\sqrt{n}}\right). \end{aligned} \tag{8.16}$$

The approximation in (8.16) uses the approximations $\exp(x) \approx 1 + x$ and $1/(1+x) \approx 1 - x$. Both these approximation assume that x is close to 0 and are appropriate in (8.16) is the number of steps n is large. Then

$$P_t = P_{m/n} = P_0 \exp\left\{\mu t + \frac{\sigma}{\sqrt{n}} \sum_{i=1}^{m} (2W_i - 1)\right\}.$$

where as before W_i is either 0 or 1 (so $2W_i - 1 = \pm 1$).

Using risk-neutral probabilities, we have $E(W_i) = q$ and $\text{Var}(W_i) = q(1-q)$ for all i. Therefore, using (8.16) and $t = m/n$ we have

$$\begin{aligned} E\left\{\frac{\sigma}{\sqrt{n}} \sum_{i=1}^{m} 2W_i - 1\right\} &= \frac{\sigma m(2q-1)}{\sqrt{n}} \approx \frac{\sigma}{\sqrt{n}} m \left(\frac{r - \mu - \sigma^2/2}{\sigma\sqrt{n}}\right) \\ &= \left(r - \mu - \frac{\sigma^2}{2}\right) \frac{m}{n} = t(r - \mu - \sigma^2/2). \end{aligned}$$

and

$$\text{Var}\left\{\frac{\sigma}{\sqrt{n}} \sum_{i=1}^{m} 2W_i - 1\right\} = \frac{4\sigma^2 m q(1-q)}{n} \approx t\sigma^2,$$

since $q \to 1/2$ so that $4q(1-q) \to 1$ as $n \to \infty$.

Therefore, in the risk-neutral world

$$P_t \approx P_0 \exp\{(r - \sigma^2/2)t + \sigma B_t\}, \tag{8.17}$$

where B_t is Brownian motion and $0 \le t \le 1$. Time could be easily extended beyond 1 by adding more steps. We will assume that this has been done.

Notice that (8.17) does NOT depend on μ, only on σ. The reason is that in the risk-neutral world, the expectation of all assets increase at rate r. The rate of increase in the real world is μ but this is irrelevant for risk-neutral calculations. Remember that risk-neutral expectations DO give the correct option price in the real world even if they do not correctly describe real world probability distributions.

If E is the exercise or strike price and T is the expiration date of a European call option, then using (8.17) the value of the option at maturity is

$$(P_T - E)_+ = \left[P_0 \exp\{(r - \sigma^2/2)T + \sigma B_T\} - E\right]_+. \tag{8.18}$$

Since $B_T \sim N(0, T)$, we can write $B_T = \sqrt{T}Z$ where $Z \sim N(0,1)$. The discounted value of (8.18) is

$$\left[P_0 \exp\left\{-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}Z\right\} - \exp(-rT)E\right]_+. \tag{8.19}$$

We will again use the principle that the price of an option is the risk-neutral expectation of the option's discounted value at expiration. By this prinicple, the call's price at time $t = 0$ is the expectation of of (8.19). Therefore,

$$C = \int \left[P_0 \exp\left\{ -\frac{\sigma^2 T}{2} + \sigma\sqrt{T}z \right\} - \exp(-rT)E \right]_+ \phi(z)dz, \qquad (8.20)$$

where ϕ is the $N(0,1)$ pdf (probability density function).

Computing this integral is not easy, but it can be done. The result is the famous Black-Scholes formula: Let S_0 be the current stock price (we have switched notation from P_0), let E be the exercise price, let r to be continuously compounded interest rate, let σ be the volatility, and let T be the expiration date of a call option. Then by evaluating the integral in (8.20) it can be shown that

$$C = \Phi(d_1)S_0 - \Phi(d_2)E\exp(-rT)$$

where Φ is the standard normal CDF,

$$d_1 = \frac{\log(S_0/E) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Example

Here's a numerical example. Suppose that $S_0 = 100$, $E = 90$, $\sigma = .4$, $r = .1$, and $T = .25$. Then

$$d_1 = \frac{\log(100/90) + \{.1 + (.4)^2/2\}(.25)}{.4\sqrt{.25}} = 0.7518$$

and

$$d_2 = d_1 - .4\sqrt{.25} = .5518.$$

Then $\Phi(d_1) = .7739$ and $\Phi(d_2) = .7095$. Also, $\exp(-rT) = \exp\{(.1)(.25)\} = .9753$. Therefore,

$$C = (.7739)(100) - (.7095)(90)(.9753) = 15.1.$$

8.12 Using the Black-Scholes formula

8.12.1 How does the option price depend on the inputs?

Figure 8.7 shows the variation in the price of a call option as the parameters change. The baseline values of the parameters are $S_0 = 100$, $E =$

$100\exp(rT)$, $T = .25$, $r = .06$, and $\sigma = .1$. The exercise price E and initial price S_0 have been chosen so that if invested at the risk-free rate, S_0 would increase to E at expiration time. There is nothing special about this choice of S_0 and E and other choices would have been possible and would be interesting to investigate. In each of the subplots in Figure 8.7, one of the parameters is varied while the others are held at baseline.

One see that the price of the call increases with σ. This makes sense since in this example, because of the way that E and S_0 have been chosen, $E = S_0\exp(rT) = E(S_T)$ where $E(S_T)$ is the risk-neutral expectation of S_T. Therefore, the expected value of $E(S_T)$ is at the money. Thus, S_T is equally likely to be in the money or out of the money. As σ increase, the likelihood that S_T is considerably larger than E also increases. As an extreme case, suppose that $\sigma = 0$. Then in the risk-neutral world $S_T = \exp(rT)S_0 = E$ and the option at expiration is at the money so its value is 0.

The value at maturity is $(S_T - E)_+$ so we expect that the price of the call will increase as S_0 increases and decrease as E increases. This is exactly the behavior seen in Figure 8.7. Also, note that the price of the call increases as either r or T increases.

8.12.2 An example — GE

Table 8.1 gives the exercise price E, month of expiration, and the price of call options on GE on February 13, 2001. This information was taken from *The Wall Street Journal*, February 14. Traded options are generally American rather than European and that is true of the options in Table 8.1. However, under the Black-Scholes theory it can be proved that the price of an American call option is identical to the price of a European call option.[6] See Section 8.12.3 for discussion of this point. Since an American call has the same price as a European call, we can use the Black-Scholes formula for European call options to price the options in Table 8.1. We will compare the Black-Scholes prices with the actual market prices.

Only the month of maturity is listed in a newspaper. However, maturities (days until expiration) can be determined as follows. An option expires on 10:59pm Central Time of the Saturday after the third Friday in the month of expiration (Hull, 1995, page 180). February 16, 2001 was the

[6]However, as discussed in Section 8.13, American and European put options will in general have different prices.

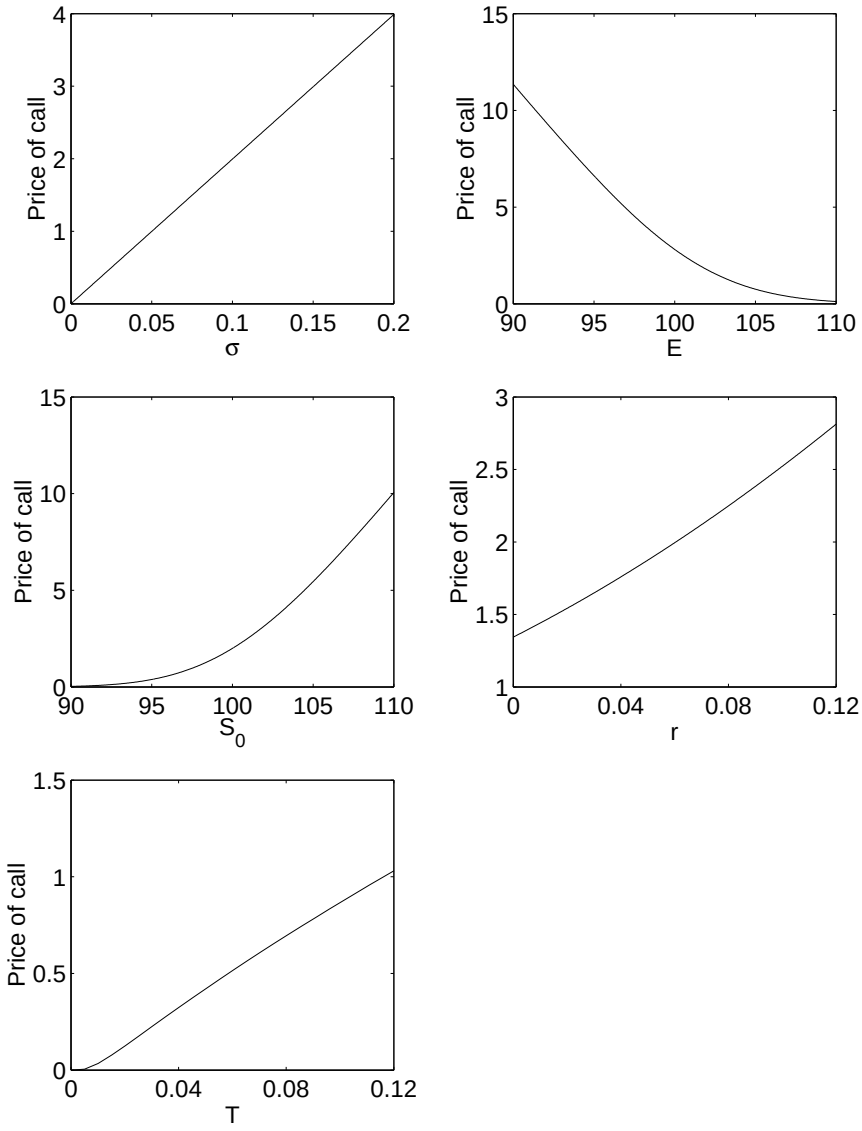

Figure 8.7: *Price of a call as a function of volatility (σ), exercise price (E), initial price (S_0), risk-free rate (r), and expiration date (T). Baseline values of the parameters are $S_0 = 100$, $E = 100\exp(rT)$, $T = .25$, $r = .06$, and $\sigma = .1$. In each subplot, all parameters except the one on the horizontal axis are fixed at baseline.*

third Friday of its month, so that on February 13, an option with a February expiration date had three trading days (and four calendar days) until expiration. Since there are returns on stocks only on trading days, $T = 3$ for options expiring in February. Similarly, on February 13 an option expiring in March had $T = 23$ trading days until expiration. Since there are 253 trading days/year, there are $253/12 \approx 21$ trading days per month. For June, I used $T = 23 + (21)(3)$ and for September I used $T = 23 + (6)(21)$.

GE closed at \$47.16 on February 13, so we use $S_0 = 47.16$.

On February 13, the 3-month T-bill rate was 4.91%. Thus, the daily rate of return on T-bills would be $r = 0.0491/253 = .00019470$, assuming that a T-bill only has a return on the 253 trading days per year; see Section 8.12.4.

I used two values of σ. The first, 0.0176, was based on daily GE return from December 1999 to December 2000. The second, 0.025, was chosen to give prices somewhat similar to the actual market prices.

E	Month of Expiration	T (in days)	Actual Price	B&S calculated price		Implied Volatility
				$\sigma = .0176$	$\sigma = .025$	
35	Sep	149	14.90	13.40	14.03	.0320
40	Sep	149	10.80	9.22	10.37	.0275
42.50	Mar	23	5.30	5.03	5.38	.0235
45	Feb	3	2.40	2.22	2.32	.0290
45	Mar	23	3.40	3.00	3.57	.0228
50	Feb	3	0.10	0.016	0.09	.0258
50	Mar	23	0.90	0.64	1.23	.0209
50	Sep	149	4.70	3.42	5.12	.0232
55	Mar	23	0.20	0.06	0.28	.0223
55	Jun	86	1.30	0.92	2.00	.0204

Table 8.1: Actual prices and prices determined by the Black-Scholes formula for options on February 13, 2001. E is the exercise price. T is the maturity.

8.12.3 Early exercise of calls is never optimal

It can be proved that early exercise of an American call option is never optimal. The reason is that at any time before the expiration date, the price of the option will be higher than the value of the option if exercised. Therefore, it is always better to sell the option rather than to exercise it early.

To see empirical evidence of this principle, consider the first option in Table 8.1. The strike price is 35 and the closing price of GE was 47.16. Thus, if the option had been exercised at the closing of the market, the option holder would have gained $(47.16 − 35) = $12.16. However, the option was selling on the market for $14.90 that day. Thus, one would gain $(14.90 − 12.16) = $2.74 more by selling the option rather than exercising it.

Similarly, the other options in Table 8.1 are worth more if sold than if exercised. The second option is worth $(47.16 − 40) = $7.16 is exercised but $10.80 if sold. The third option is worth $(47.16 − 42.5) = $4.66 if exercised but $5.30 if exercised.

Since it is never optimal to exercise an American call option early, the ability to exercise an American call early is not worth anything. This is why American calls are equal in value to European calls with the same exercise price and expiration date.

8.12.4 Are there returns on non-trading days?

We have assumed that there are no returns on non-trading days. For T-bills, this assumption is justified by the way we calculated the daily interest rate. We took the daily rate to be the annual rate divided by 253 on trading days and 0 on non-trading days. If instead we took the daily rate to be the annual rate divided by 365 on every calendar day, then the interest on T-Bills over a year, or a quarter, would be the same.

A stock price is unchanged over a non-trading day. However, the efficient market theory says that stock prices change due to new information. Thus, we might expect that there is a return on a stock over a weekend or holiday but it is realized only when the market reopens. If this were true, then returns from Friday to Monday would be more volatile than returns over a single trading day. Empirical evidence fails to find such an effect.

A reason why returns over weekends are not overly volatile might be that there is little business news over a weekend. However, this does not seem to be the explanation why there is not excess volatility over a weekend. In 1968, the NYSE was closed for a series of Wednesday. Of course, other businesses were open on these Wednesdays so there was the usual amount of business news during the Wednesdays when the NYSE was closed. For this reason, one would expect increased volatility for Tuesday to Thursday price changes on weeks with a Wednesday closing compared to, say, Tuesday to Wednesday price changes on weeks without a Wednesday market closing. However, no such effect has been detected (French and Roll, 1986).

Trading appears to generate volatility by itself. Traders react to each other. Stock prices react to both trading "noise" and to new information. Short term volatility might be mostly due to noise trading.

8.12.5 Implied volatility

Given the exercise price, current price, and maturity of an option and given the risk-free rate, there is some value of σ that makes the price determined by the Black-Scholes formula equal to the current market price. This value of σ is called the *implied volatility*. One might think of implied volatility as the amount of volatility the market believes to exist currently.

How does one determine the implied volatility? The Black-Scholes formula gives price as a function of σ with all other parameters held fixed. What we need is the inverse of this function, that is, σ as a function of the option price. Unfortunately, there is no formula for the inverse function. The function exists, of course, but there is no explicit formula for it. However, using interpolation one can invert the Black-Scholes formula to get σ as a function of price. Figure 8.8 shows how this could be done for the third option in Table 8.1 The implied volatility in Figure 8.8 is 0.0235 and was determined by MATLAB's interpolation function, interp1.m. The implied volatilities of the other options in Table 8.1 were determined in the same manner.

Notice that the implied volatilities are substantially higher than 0.0176, the average volatility over the previous year. However, there is evidence that volatility of GE was increasing at the end of last year; see the estimated volatility in Figure 3.5. In that figure, volatility is estimated from December 15, 1999 to December 15, 2000. Volatility is highest at the end of this period and shows some sign of continuing to increase. The estimated volatility on December 15, 2000 was 0.023, which is similar to the implied volatilities in Table 8.1. It would be worthwhile to re-estimate volatility with data from December 15, 2000 to February 13, 2001. It may be that the implied volatilities in Table 8.1 are similar to the observed volatility in early 2001.

The implied volatilities also vary somewhat among themselves. One reason for this variation is that the option prices and closing price of GE stock are not concurrent. Rather, each price is for the last trade of the day for that option or for the stock. This lack of concurrence introduces some error into pricing by the Black-Scholes formula and therefore into the implied

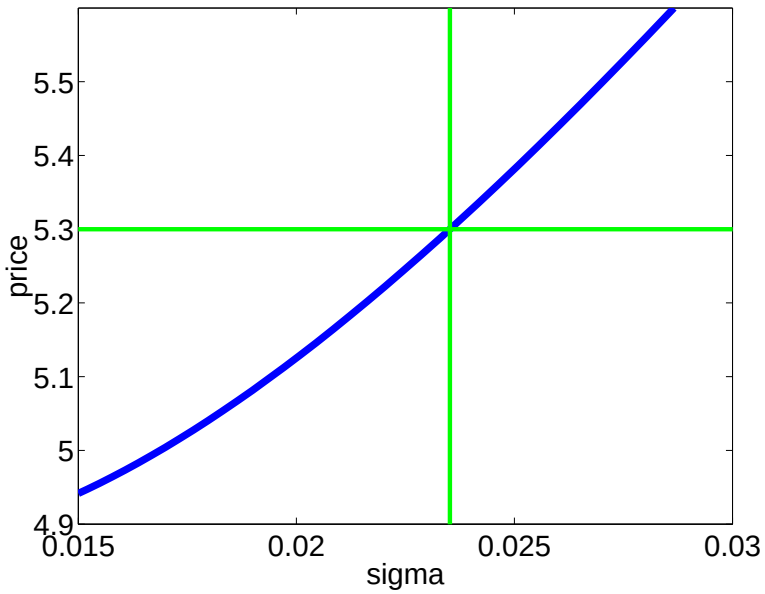

Figure 8.8: *Calculating the volatility implied by the option with an exercise price of $42.50 expiring in March 2001. The price was $5.30 on February 13, 2001. The blue curve is the price given by the Black-Scholes formula as a function of σ. The horizontal line is drawn where price is $5.30. This line intersects the curve at $\sigma = .0242$. This value of σ is the volatility implied by the option's price.*

volatitities. Another problem with these prices is that the Black-Scholes formula assumes that the stock pays no dividends, but GE does pay dividends.[7]

8.13 Puts

Recall that a put option gives one the right to sell a certain number of shares of a certain stock at the exercise price. The pricing of puts is similar to the pricing of calls, but as we will see in this section, there are some differences.

8.13.1 Pricing puts by binomial trees

Put options can be priced by binomial trees in the same way that call options are priced. Figure 8.9 shows a two-step binomial tree where the stock price starts at \$100 and increases or decreases by 20% at each step. Assume that the interest rate is 5% compounded continuously and that the strike price of the put is \$110.

In this example, European and American puts do NOT have the same price at all nodes. We will start with a European put and then see how an American put differs.

At each step,

$$q = \frac{\exp(.05) - .8}{1.2 - .8} = .6282.$$

The value of a put after two steps is $(110 - S)_+$ where S is the price of the stock after two steps. Thus the put is worth \$46, \$14, and \$0 at nodes 4, 5, and 6 respectively. Therefore, the price of the option at node 3 is

$$e^{-.05}\{(q)(0) + (1-q)(14)\} = e^{.05}\{(.6282)(0) + (.3718)(14)\} = 4.91.$$

The price of the option at node 2 is

$$e^{-.05}\{(q)(14) + (1-q)(46)\} = 24.63.$$

Finally the price of the put at node 1 is

$$e^{-.05}\{(q)(4.91) + (1-q)(24.63)\} = 11.65.$$

[7]Modifications of the formula to accommodate dividend payments are possible, but we will not pursue that topic here.

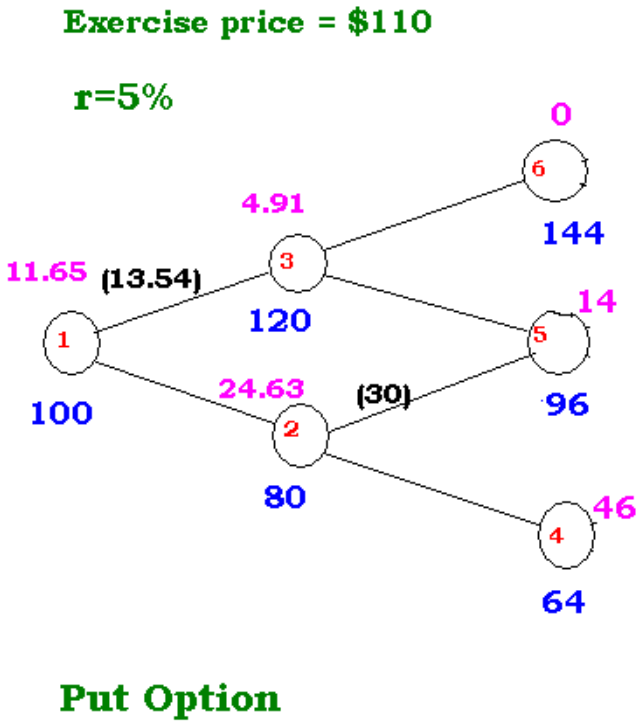

Figure 8.9: *Pricing a put option. The stock price is in blue and the price of a European put option is in magenta. The price of an Amercian put option is shown in black with parentheses when it differs from the price of a European put.*

Now consider an American option. At nodes 4, 5, and 6 we have reached the expiration time so that the American option has the same value as the European option.

At node 3 the European option is worth \$4.91. At this node, should we exercise the American option early? Clearly not, since the strike price (\$110) is less than the stock price (\$120). Since early exercise is suboptimal at node 3, the American option is equivalent to the European option at this node and both options are worth \$4.91.

At node 2 the European option is worth \$24.63. The American option can be exercised to earn (\$110 − \$80) = \$30. Therefore, the American option should be exercised early since early exercise earns \$30 while holding the option is worth only \$24.63. Thus, at node 2 the European option is worth \$24.63 but the American option is worth \$30.

At node 1, the American option is worth

$$e^{-.05}\{(q)(4.91) + (1-q)(30)\} = 13.65,$$

which is more than \$11.65, the value of the European option at node 1. The American option should NOT be exercised early at node 1 since that would earn only \$10. However, the American option is worth more than the European option at node 1 because the American can be exercised early at node 2 should the stock move down at node 1.

8.13.2 Why are puts different than calls?

We saw that in the Black-Scholes model where changes in price are proportional to the current price, it is never optimal to exercise an American call early. Puts are different. In the Black-Scholes model, early exercise of a call may be optimal. In the binomial model example just given, prices changes are proportional to current prices as in the Black-Scholes model, and in the binomial model early exercise of a put is again optimal at some nodes.

So why are puts different than calls? The basic idea is this. A put increases in value as the stock price *decreases*. As the stock price decreases, the size of further price changes also decreases. At some point we are in the range of diminishing returns. We expect further decreases in the stock price to be so small that the put will increase in value at less than the risk-free rate. Therefore, it is better to exercise the option and invest the profits in a risk-free asset.

With calls, everything is reversed. A call increases in value as the stock price increases. As the stock price increases, so does the size of future price

changes. The expected returns on the call (expectations are with respect to the risk-neutral measure, of course) are always greater than the risk-free rate of return.

8.13.3 Put-call parity

It is possible, of course, to derive the Black-Scholes formula for a European put option by the same reasoning used to price a call. However, this work can be avoided since there is a simple formula relating the price of a European put to that of a call:

$$P = C + e^{-rT}E - S_0, \tag{8.21}$$

where P and C are the prices of a put and of a call, both with expiration date T and exercise price E. Here, the stock price is S_0 and r is the continuously compounded risk-free rate. Thus, the price of a put is simply the price of the call plus $(e^{-rT}E - S_0)$.

Equation (8.21) is derived by a simple arbitrage argument. Consider two portfolios. The first portfolio holds one call and Ee^{-rT} dollars in the risk-free asset. Its payoff at time T is E, the value of the risk-free asset, plus the value of the call, which is $(S_T - E)_+$. Therefore, its payoff is E if $S_T < E$ and S_T if $S_T > E$. In other words, the payoff is either E or S_T, whichever is larger.

The second portfolio holds a put and one share of stock. Its payoff at time T is S_T if $S_T \geq E$ so that the put is not exercised. If $S_T < E$, then the put is exercised and the stock is sold for a payoff of E. Thus, the payoff is E or S_T, whichever is larger, which is the same payoff as the first portfolio.

Since the two portfolios have the same payoff for all values of S_T, their initial values at time 0 must be equal to avoid arbitrage. Thus,

$$C + e^{-rT}E = P + S_0,$$

which can be rearranged to yield equation (8.21).[8]

Relationship (8.21) holds only for European options. European calls have the same price as European calls so that the right hand side of (8.21)

[8] As usual in these notes, we are assuming that the stock pays no dividend, at least not during the lifetime of the two options. If there are dividends, then a simple adjustment of formula (8.21) is needed. The reason the adjustment is needed is that the two portfolios will no longer have exactly the same payoff. One can see that the first portfolio which holds the stock will receive a dividend and so receive a higher payoff than the second portfolio which holds no stock and therefore will not receive the dividend.

is the same for European and American options. American puts are worth more than European puts, so the left hand side of (8.21) is larger for American than for European puts. Thus, (8.21) becomes

$$P > C + e^{-rT}E - S_0, \tag{8.22}$$

for American options, and clearly (8.22) does not tell us the price of an American put.

8.14 The evolution of option prices

As time passes the price of an option changes with the changing stock price and the decreasing about of time until the expiration date. We will assume that r and σ are constant, though in the real financial world these could change too. The Black-Scholes formula remains in effect and can be used to update the price of an option. Suppose that $t = 0$ is when the option was written and $t = T$ is the expiration date. Consider a time point t such that $0 < t < T$. Then the Black-Scholes formula can be used with S_0 in the formula set equal to S_t and T in the formula set equal to $T - t$.

Figure 8.10 illustrates the evolution of option prices for two simulations of the geometric Brownian motion process of the stock price. Here $T = 1$, $\sigma = .1$, $r = .06$, $S_0 = 100$, and $E = 100$ for both the put and the call. In one case the call was in the money at expiration, while in the second case it was the put that was in the money.

Notice that around $t = .18$ the stock price is around 110 in the red simulation but the put is still worth something, since there is still plenty of time for the price to go down. Around $t = 1$ the stock price of the blue simulation is around 110 but the value of the put is essentially 0; now there is too little time for the put to go in the money (the risk-neutral probability is not 0, but almost 0).

8.15 Intrinsic value and time value

The intrinsic value of a call is $(S_0 - E)_+$, the payoff one would obtain for immediate exercise of the option (which would be possible only for an American option). The intrinsic value is always less than the price, so immediate exercise is never optimal. The difference between the intrinsic value and

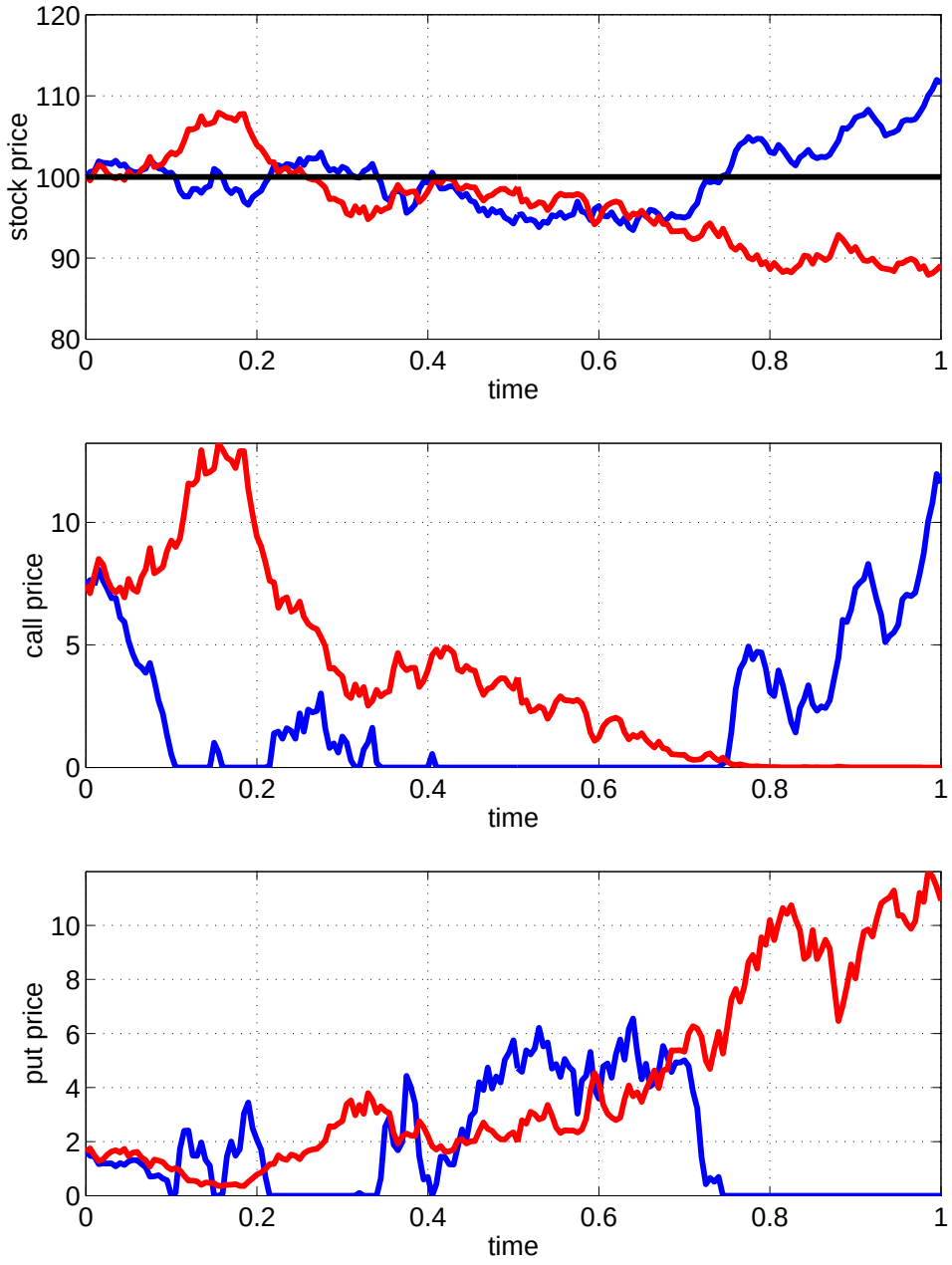

Figure 8.10: *Evolution of option prices. The stock price is a geometric Brownian motion. Two independent simulations of the stock price are shown and color coded. Here* $T = 1$, $\sigma = .1$, $r = .06$, $S_0 = 100$, *and* $E = 100$ *for both the put and the call. In the blue and red simulations the call, respectively, put are in the money at the expiration date.*

the price is called the time value of the option. Time value has two components. The first is a volatility component. The stock price could drop between now and the expiration date; by waiting until the last moment, one can avoid exercising the option when $S_T < E$. The second component is the time value of *money*. If you do exercise the option, it is best to wait until time T so that you delay payment of the exercise price.

The adjusted intrinsic value is $(S_0 - e^{-rT}E)_+$ which is greater than the intrinsic value $(S_0 - E)_+$. The difference between the price and the adjusted intrinsic value is the volatility component of the time value of the option. As $S_0 \to \infty$, the price converges to the adjusted intrinsic value and the volatility component converges to 0. The reason this happens is that as $S_0 \to \infty$ you become sure that the option will be in the money at the expiration date.

Figure 8.11 shows the price, intrinsic value, and adjusted intrinsic value of a call option when $S_0 = 100$, $E = 100$, $T = .25$, $r = 0.06$, and $\sigma = 0.1$

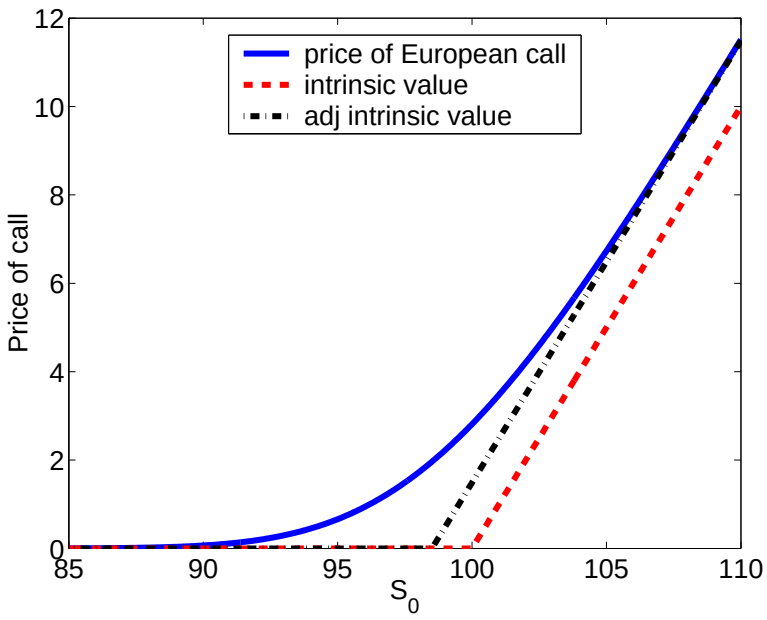

Figure 8.11: *Price (for European or American option), intrinsic value, and adjusted intrinsic value of a call option. The intrinsic value is the payoff if one exercises early. Here* $E = 100$, $T = .25$, $r = 0.06$, *and* $\sigma = 0.1$.

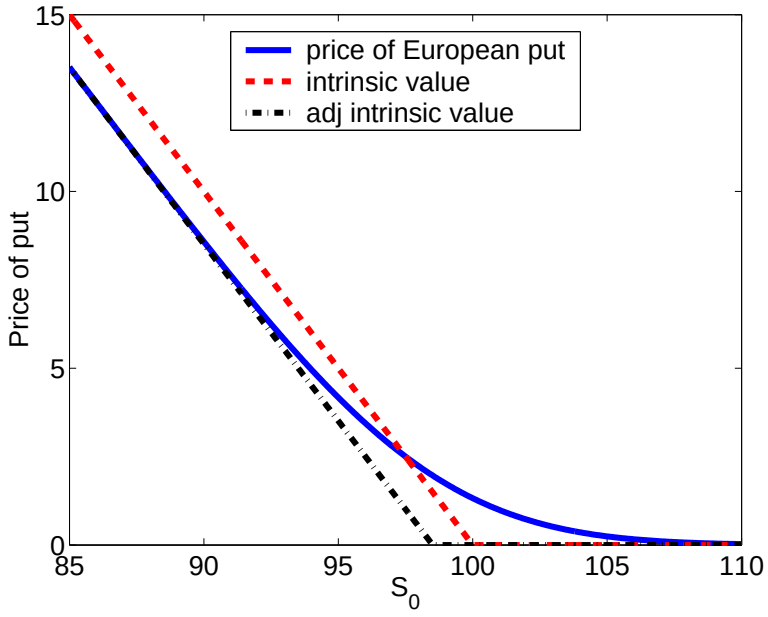

Figure 8.12: *Price (for European option), intrinsic value, and adjusted intrinsic value of a put option. The intrinsic value is the payoff if one exercises early. The price of an American put would be either the price of the European put of the intrinsic value, whichever is larger. Here* $S_0 = 100$, $E = 100$, $T = .25$, $r = 0.06$, *and* $\sigma = 0.1$.

The intrinsic value of put is $(E - S_0)_+$, which again is the the payoff one would obtain for immediate exercise of the option, if that is possible (American option). The intrinsic value is sometimes greater than the price, in which case immediate exercise is optimal.

The adjusted intrinsic value is $(e^{-rT}E - S_0)_+$. As $S_0 \to 0$, the likelihood that the option will be in the money at the expiration date increase to 1 and the price converges to the adjusted intrinsic value.

Figure 8.12 shows the price, intrinsic value, and adjusted intrinsic value of a put option when $S_0 = 100$, $E = 100$, $T = .25$, $r = 0.06$, and $\sigma = 0.1$

8.16 Black, Scholes, and Merton

This section is based on chapter 11 of Bernstein's (1992) book *Capital Ideas*.

Fischer Black graduated in 1959 from Harvard with a degree in physics. In 1964 he received a PhD in applied mathematics from Harvard where he studied operations research, computer design, and artifical intelligence. He never took a course in either finance or economics.

Finding his doctorial studies at bit too abstract, he went to work at Arthur D. Little where he became acquainted with the CAPM. He found this subject so fascinating that he moved into finance. At ADL, Black tried to apply the CAPM to the pricing of warrants, which are much like options. Bernstein (1992) quotes Black as recalling

> I applied the Capital Asset Pricing Model to every moment in a warrant's life, for every possible stock price and warrant value ... . I stared at the differential equation for many, many months. I made hundreds of silly mistakes that led me down blind alleys. Nothing worked ...
>
> [The calculations revealed that] the warrant value did not depend on the stock's expected return, or on any other asset's expected return. That fascinate me. ... Then Myron Scholes and I started working together.

Scholes received a bacheler's degree from McMaster's University in Ontario in 1962, earned a doctorate in finance from Chicago, and then took a teaching job at MIT. When Scholes meet Black he too was working intensely on warrant pricing by the CAPM. Realizing that they were working on the same problem, they began a collaboration that proved to be very fruitful.

Black and Scholes came to understand that the expected return on a stock or option had no effect of what the current price of the option should be. With this insight and building on the CAPM, they arrived at the option equation and derived the formula for the option price.

In 1970, Scholes described his work with Black on options pricing to Robert C. Merton. Merton had studied engineering mathematics at Columbia and then Cal Tech. He developed an interest in economics and planned to study that subject in graduate school. His lack of formal training in economics put off many graduate schools, but MIT offered him a fellowship where he worked under the direction of Paul Samuelson.

Merton developed the "intertemportal capital asset pricing model" that converted the CAPM from a static model describing the market for a single discrete holding period to a model for finance in continuous time. Merton

realized that Ito's stochastic calculus was a goldmine for someone working on finance theory in continuous time. In the preface to his book, "Continuous-Time Finance," Merton has written

> The mathematics of the continuous-time model contains some of the most beautiful applications of probability and optimization theory. But, of course, not all that is beautiful in science need also be practical. And surely, not all that is practical in science is beautiful. Here we have both.

Merton developed a much more elegant derivation of the Black-Scholes formula, a derivation based on an arbitrage argument. Black has said "A key part of the options paper I wrote with Myron Scholes was the arbitrage argument for deriving the formula. Bob gave us that argument. It should probably be called the Black-Merton-Scholes paper."

In 1997, Merton shared the Nobel Prize in Economics with Scholes. Sadly, Black had died at a young age and could not share the prize, since the Nobel Prize cannot be awarded posthumously. Merton has been called "the Isaac Newton of modern finance."

8.17 Summary

- An option gives the holder the right but not the obligation to do something, for example, to purchase a certain amount of a certain stock at a fixed price within a certain time frame.
- A call option gives one the right to purchase (call in) an asset. A put gives one the right to sell (put away) an asset.
- European options can be exercised only at their expiration date. American options can be exercised on or before their expiration date.
- Arbitrage is making a guaranteed profit without investing capital.
- Arbitrage pricing means determining the unique price of a financial instrument that guarantees that the market is free of arbitrage opportunities.
- Options can be priced by arbitrage using binomial trees.
- The "measure" of a binomial tree model or other stochastic process model gives the set of path probabilities of that model.

- There exists a risk-neutral measure such that expected prices calculating with respect to this measure are equal to arbitrage determined prices.
- In a binomial tree model with price changes proportional to the current price, as the number of steps increases the limit process is a geometric Brownian motion and the price of the option in the limit is given by the Black-Scholes formula.
- To price an option by the Black-Scholes formula, one needs an estimate of the stock price's volatility. This can be obtained from historical data. Conversely, the implied volatility of a stock is the volatility which makes the actual market price equal to the price given by the Black-Scholes formula.
- Within the Black-Scholes model, the early exercise of calls is never optimal but the early exercise of puts is sometimes optimal. Therefore, European and American calls have equal prices, but American puts are generally worth more than European puts.
- Put-call parity is the relationship

$$P = C + e^{-rT}E - S_0$$

between P, the price of a European put, and C, the price of a European call. It is assumed that both have exercise price E and expiration date T. S_0 is the price of the stock.

8.18 References

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Chapter 9

GARCH models: 4/5/02

9.1 Introduction

Despite the popularity of ARMA models, they have a significant limitation, namely, that they assume a constant volatility. In finance, where correct specification of volatility is of the utmost importance, this can be a severe limitation. In this chapter we look at GARCH time series models that have randomly varying volatility. These models are becoming widely used in econometrics and finance.

ARMA models are used to model the conditional expectation of the current observation, Y_t, of a process given the past observations. ARMA models do this by writing Y_t as a linear function of the past plus a white noise term. ARMA models also allow us to predict future observations given the past and present. The prediction of Y_{t+1} given $Y_t, Y_{t-1} \ldots$ is simply the conditional expectation of Y_{t+1} given $Y_t, Y_{t-1} \ldots$.

However, ARMA models have rather boring conditional variances—the conditional variance of Y_t given the past is always a constant. What does this mean for, say, modeling stock returns? Suppose we have noticed that recent daily returns have been unusually volatile. We might suppose that tomorrow's return will also be more variable than usual. However, if we are modeling returns as an ARMA process, we cannot capture this type of behavior because the conditional variance is constant. So we need better time series models if we want to model the nonconstant volatility often seen in financial time series.

In this chapter we will study models of nonconstant volatility. ARCH is an acronym meaning AutoRegressive Conditional Heteroscedasticity.[1] In

[1]Heteroscedasticity is a fancy way of saying non-constant variance. Homoscedasticity

ARCH models the conditional variance has a structure very similar to the structure of the conditional expectation in an AR model. We will first study the ARCH(1) model, which is similar to an AR(1) model. Then we will look at ARCH(p) models which are analogous to AR(p) models. Finally, we will look at GARCH (Generalized ARCH) models which model conditional variances much like the conditional expectation of an ARMA model.

9.2 Modeling conditional means and variances

Before looking at GARCH models, we will study some general principles on how one models non-constant conditional variance.

We will first look at modeling with a *constant* conditional variance. Suppose that $\text{Var}(Y_t|X_{1,t},\ldots,X_{p,t}) = \sigma^2$, a constant. Then, the general form for the regression of Y_t on $X_{1.t},\ldots,X_{p,t}$ is

$$Y_t = f(X_{1,t},\ldots,X_{p,t}) + \epsilon_t \tag{9.1}$$

where ϵ_t has expectation equal to 0 and a constant variance σ^2. The function f is the conditional expectation of Y_t given $X_{1,t},\ldots,X_{p,t}$. To appreciate this fact, notice that if we take the conditional (given the $X_{i,t}$ values) expectation of (9.1), $f(X_{1,t},\ldots,X_{p,t})$ is treated as a constant and the conditional expectation of ϵ_t is 0. Moreover, the conditional variance is simply the variance of ϵ_t, that is, σ^2. Frequently, f is linear so that

$$f(X_{1,t},\ldots,X_{p,t}) = \beta_0 + \beta_1 X_{1,t} + \cdots + \beta_p X_{p,t}.$$

Principle: To model the conditional mean of Y_t given $X_{1.t},\ldots,X_{p,t}$, write Y_t as the conditional mean *plus* white noise.

Equation (9.1) can be modified to allow a nonconstant conditional variance. Let $\sigma^2(X_{1,t},\ldots,X_{p,t})$ be the conditional variance of Y_t given $X_{1,t},\ldots,X_{p,t}$. Then the model

$$Y_t = f(X_{1,t},\ldots,X_{p,t}) + \sigma(X_{1,t},\ldots,X_{p,t})\,\epsilon_t \tag{9.2}$$

gives the correct conditional mean and variance.

Principle: To allow a nonconstant conditional variance in the model, *multiply* the white noise term by the conditional standard deviation. This product is added to the conditional mean as in the previous principle.

The function $\sigma(X_{1,t},\ldots,X_{p,t})$ must be non-negative since it is a standard deviation. If the function $\sigma(\cdot)$ is linear, then its coefficients must be

means constant variance. Alternate spellings are heteroskedasticity and homoskedasticity.

constrained to ensure non-negativity. Modeling non-constant conditional variances in regression is treated in depth in the book by Carroll and Ruppert (1988). Models for conditional variances are often called "variance function models." The GARCH models of this chapter are a special class of variance function models.

9.3 ARCH(1) processes

Let $\epsilon_1, \epsilon_2, \ldots$ be Gaussian white noise with unit variance, that is, let this process be independent N(0,1). Then

$$E(\epsilon_t|\epsilon_{t-1}, \ldots) = 0,$$

and

$$\text{Var}(\epsilon_t|\epsilon_{t-1}, \ldots) = 1. \tag{9.3}$$

Property (9.3) is called *conditional homoscedasticity*.

The process a_t is an ARCH(1) process if

$$a_t = \epsilon_t \sqrt{\alpha_0 + \alpha_1 a_{t-1}^2}. \tag{9.4}$$

We require that $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$ because a standard deviation cannot be negative. It is also required that $\alpha_1 < 1$ in order for a_t to be stationary with a finite variance. If $\alpha_1 = 1$ then a_t is stationary but its variance is ∞; see Section 9.10 below. Equation (9.4) can be written as

$$a_t^2 = (\alpha_0 + \alpha_1 a_{t-1}^2)\, \epsilon_t^2,$$

which is very much like an AR(1) but in a_t^2, not a_t and with multiplicative noise with a mean of 1 rather than additive noise with a mean of 0 in fact, the ARCH(1) model induces an ACF in a_t^2 that is the same as an AR(1)'s ACF.

Define

$$\sigma_t^2 = \text{Var}(a_t|a_{t-1}, \ldots)$$

to be the conditional variance of a_t given past values. Since ϵ_t is independent of a_{t-1} and $E(\epsilon_t^2) = \text{Var}(\epsilon_t) = 1$

$$E(a_t|a_{t-1}, \ldots) = 0, \tag{9.5}$$

and

$$\begin{aligned} \sigma_t^2 &= E\{(\alpha_0 + \alpha_1 a_{t-1}^2)\, \epsilon_t^2 | a_{t-1}, a_{t-2}, \ldots\} \\ &= (\alpha_0 + \alpha_1 a_{t-1}^2) E\{\epsilon_t^2 | a_{t-1}, a_{t-2}, \ldots\} \\ &= \alpha_0 + \alpha_1 a_{t-1}^2. \end{aligned} \tag{9.6}$$

Understanding equation (9.6) is crucial to understanding how GARCH processes work. This equation shows that if a_{t-1} has an unusually large deviation from its expectation of 0, so that a_{t-1}^2 is large, then the conditional variance of a_t is larger than usual. Therefore, a_t is also expected to have an unusually large deviation from its mean of 0. This volatility will propagate since a_t having a large deviation makes σ_{t+1}^2 large so that a_{t+1} will tend to be large. Similarly, if $|a_{t-1}|$ is unusually small, then σ_t^2 will be small, and a_t is expected to also be small, etc. Because of this behavior, unusual volatility in a_t tends to persist, though not forever. The conditional variance tends to revert to the unconditional variance provided that $\alpha_1 < 1$ so that the process is stationary with a finite variance.

The unconditional, i.e., marginal, variance of a_t denoted by $\gamma_a(0)$ is gotten by taking expectations in (9.5) which give us

$$\gamma_a(0) = \alpha_0 + \alpha_1 \gamma_a(0).$$

This equation has a positive solution if $\alpha_1 < 1$:

$$\gamma_a(0) = \alpha_0/(1 - \alpha_1).$$

If $\alpha_1 \geq 1$ then $\gamma_a(0)$ is infinite. It turns out that a_t is stationary nonetheless. The integrated GARCH model (I-GARCH) has $\alpha_1 = 1$ and is discussed in Section 9.10.

Straightforward calculations using (9.6) show that the ACF of a_t is

$$\rho_a(h) = 0 \qquad \text{if} \quad h \neq 0.$$

In fact, any process such that the conditional expectation of the present observation given the past is constant is an uncorrelated process. In introductory statistics courses, it is often mentioned that independence implies zero correlation but not vice versa. A process, such as the GARCH processes, where the conditional mean is constant but the conditional variance is non-constant is a good example of a process that is uncorrelated but not independent. The dependence of the conditional variance on the past is the reason the process is not independent. The independence of the conditional mean on the past is the reason that the process is uncorrelated.

Although a_t is uncorrelated just like the white noise process ϵ_t, the process a_t^2 has a more interesting ACF: if $\alpha_1 < 1$ then

$$\rho_{a^2}(h) = \alpha_1^{|h|}, \qquad \forall \quad h.$$

If $\alpha_1 \geq 1$, then a_t^2 is nonstationary, so of course it does not have an ACF.

9.3.1 Example

A simulated ARCH(1) process is shown in Figure 9.1. The top-left panel shows the independent white noise process, ϵ_t. The top right panel shows $\sigma_t = \sqrt{1 + .95a_{t-1}^2}$, the conditional standard deviation process. The bottom left panel shows $a_t = \sigma_t \epsilon_t$, the ARCH(1) process. As discussed in the next section, an ARCH(1) process can be used as the noise term of an AR(1) process. This is shown in the bottom right panel. The AR(1) parameters are $\mu = .1$ and $\phi = .8$.

The variance of a_t is $\gamma_a(0) = 1/(1 - .95) = 20$ so the standard deviation is $\sqrt{20} = 4.47$.

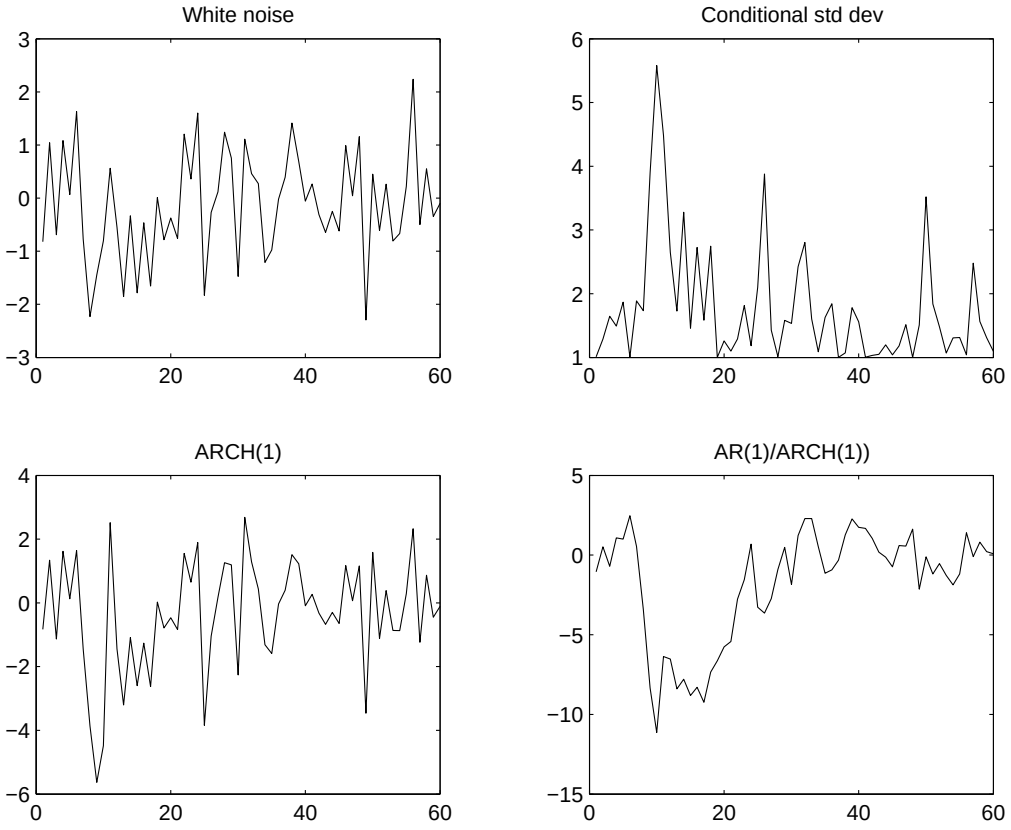

Figure 9.1: *Simulation of 60 observations from an ARCH(1) process and an AR(1)/ARCH(1) process. The parameters are* $\alpha_0 = 1$, $\alpha_1 = .95$, $\mu = .1$, *and* $\phi = .8$.

The processes were started out all equal to 0 and simulated for 70 observation. The first 10 observations were treated as a burn-in period where the process was converging to its stationary distribution. In the figure, only the last 60 observations are plotted.

The white noise process in the top left panel is normally distributed and has a standard deviation of 1, so it will be less that 2 in absolute value about 95% of the time. Notice that just before $t = 10$, the process is a little less than -2 which is a somewhat large deviation from the mean of 0. This deviation causes the conditional standard deviation (σ_t) shown in the top right panel to increase and this increase persists for about 10 observations though it slowly decays. The result is that the ARCH(1) process exhibits more volatility than usual when t is between 10 and 15.

Figure 9.2 shows a simulation of 600 observations from the same processes as in Figure 9.1. A normal probability plot of a_t is also included. Notice that this ARCH(1) exhibits extreme non-normality. This is typical of ARCH processes. Conditionally they are normal with a nonconstant variance, but their marginal distribution is non-normal with a constant variance.

9.4 The AR(1)/ARCH(1) model

As we have seen, an AR(1) has a nonconstant conditional mean but a constant conditional variance, while an ARCH(1) process is just the opposite. If we think that both the conditional mean and variance of a process will depend on the past then we need the features of both the AR and ARCH models. Can we combine the two models? Fortunately, the answer is "yes." In fact, we can combine any ARMA model with any GARCH model. In this section we start simple and combine an AR(1) model with an ARCH(1) model.

Let a_t be an ARCH(1) process and suppose that

$$u_t - \mu = \phi(u_{t-1} - \mu) + a_t.$$

u_t looks like an AR(1) process, except that the noise term is not independent white noise but rather an ARCH(1) process.

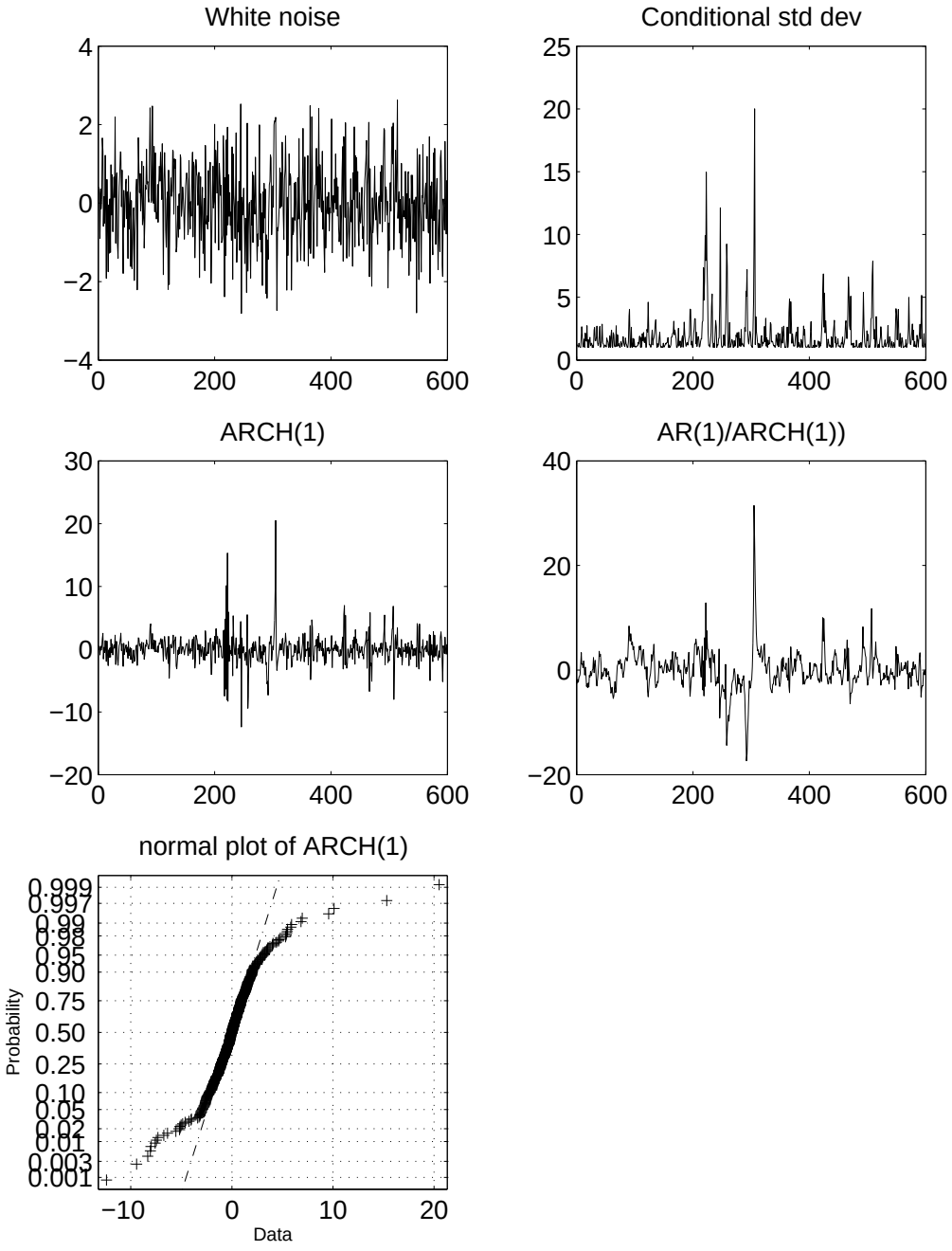

Figure 9.2: *Simulation of 600 observations from an ARCH(1) process and an AR(1)/ARCH(1) process. The parameters are* $\alpha_0 = 1$, $\alpha_1 = .95$, $\mu = .1$, *and* $\phi = .8$.

Although a_t is not independent white noise, we saw in the last section that it is an uncorrelated process; a_t has the same ACF as independent white noise. Therefore, u_t has the same ACF as an AR(1) process:

$$\rho_u(h) = \phi^{|h|} \qquad \forall \quad h.$$

Moreover, a_t^2 has the ARCH(1) ACF:

$$\rho_{a^2}(h) = \alpha_1^{|h|} \qquad \forall \quad h.$$

We need to assume that both $|\phi| < 1$ and $\alpha_1 < 1$ in order for u to be stationary with a finite variance. Of course, $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$ and also assumed.

The process u_t is such that its conditional mean and variance, given the past, are both nonconstant so a wide variety of real time series can be modeled.

Example

A simulation of an AR(1)/ARCH(1) process is shown in the bottom right panel of Figure 9.1. Notice that when the ARCH(1) noise term in the bottom left panel is more volatile, then the AR(1)/ARCH(1) process moves more rapidly.

9.5 ARCH(q) models

As before, let ϵ_t be Gaussian white noise with unit variance. Then a_t is an ARCH(q) process if

$$a_t = \sigma_t \epsilon_t$$

where

$$\sigma_t = \sqrt{\alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2}$$

is the conditional standard deviation of a_t given the past values $a_{t-1}, a_{t-2}, \ldots$ of this process. Like an ARCH(1) process, an ARCH(q) process is uncorrelated and has a constant mean (both conditional and unconditional) and a constant unconditional variance, but its conditional variance is nonconstant. In fact, the ACF of a_t^2 is the same as the ACF on an AR(q) process.

9.6 GARCH(p, q) models

The GARCH(p, q) model is

$$a_t = \epsilon_t \sigma_t$$

where

$$\sigma_t = \sqrt{\alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2}.$$

The process a_t is uncorrelated with a stationary mean and variance and a_t^2 has an ACF like an ARMA process.

A very general time series model lets a_t be GARCH(p_G, q_G) and uses a_t as the noise term in an ARIMA(p_A, d, q_A) model.[2]

GARCH models include ARCH models as a special case, and we will use the term "GARCH" to refer to both ARCH and GARCH models.

Figure 9.3 is a simulation of 600 observations from a GARCH(1,1) process and from a AR(1)/ GARCH(1,1) process. The GARCH parameters are $\alpha_0 = 1$, $\alpha_1 = .08$, $\beta_1 = .9$. The large value of β_1 give the conditional standard deviation process a long-term memory. Notice that the conditional standard deviation is less "bursty" than for an ARCH(1) process such as in Figure 9.2.

9.7 Heavy-tailed distributions

Researchers have long noticed that stock returns have "heavy-tailed" or "outlier-prone" probability distributions. This means that they have more extreme outliers than expected from a normal distribution. The reason for the outliers may be that the conditional variance is not constant, and the outliers occur when the variance is large. In fact, GARCH processes exhibit heavy-tails. Therefore, when we use GARCH models in finance we can model both the conditional heteroscedasticity and the heavy-tailed distributions of financial market data. This makes GARCH models especially useful in financial applications.

To understand how a non-constant variance induces outliers, we look at a simple case. Consider a distribution which is 90% $N(0, 1)$ and 10%

[2]We use subscripts on p and q to distinguish between the GARCH (G) and ARIMA (A) parameters.

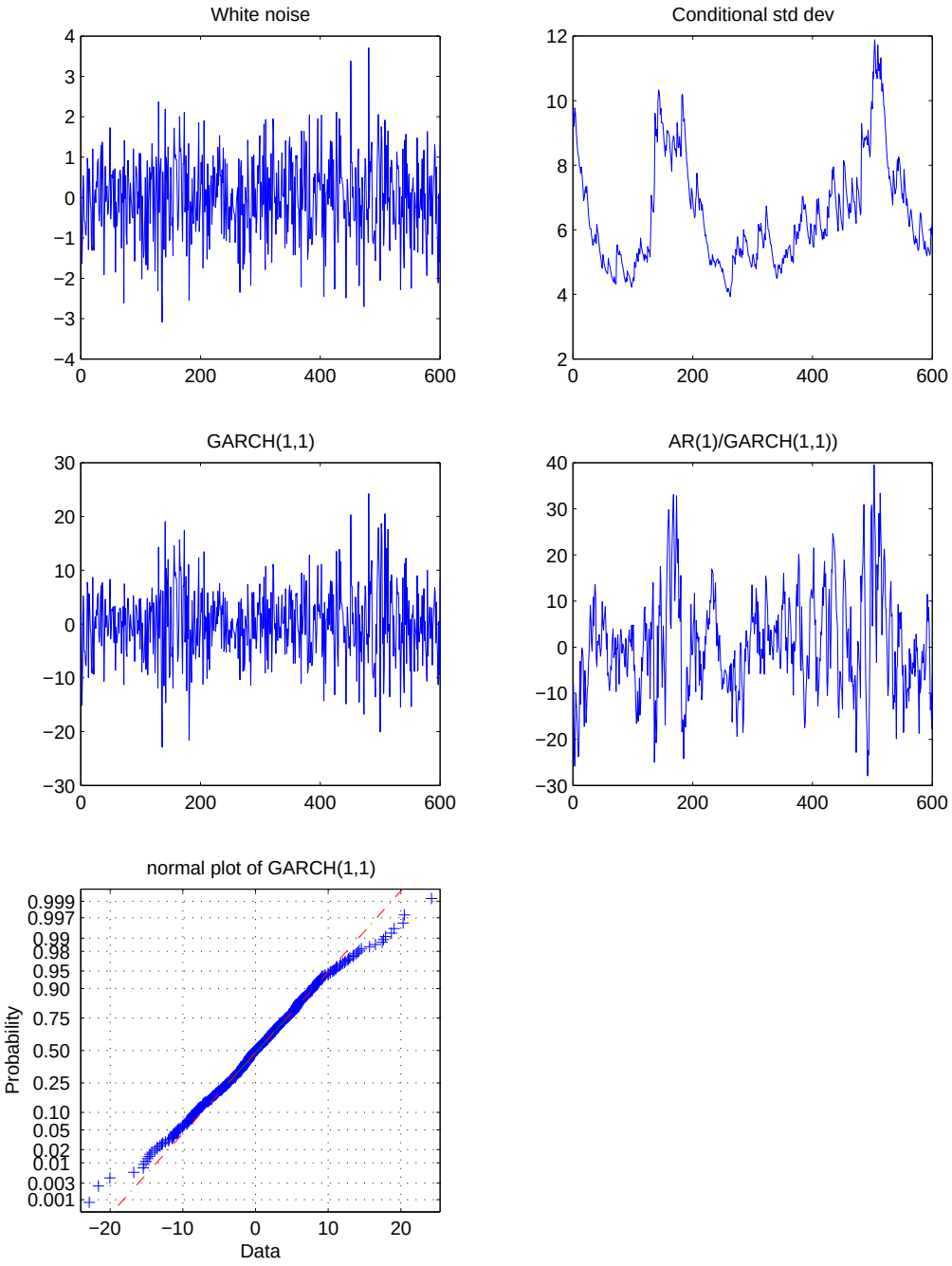

Figure 9.3: *Simulation GARCH(1,1) and AR(1)/GARCH(1,1) processes. The parameters are* $\alpha_0 = 1$, $\alpha_1 = .08$, $\beta_1 = .9$, *and* $\phi = .8$.

$N(0, 25)$. This is an example of a "normal mixture" distribution since it is a mixture of two different normal distributions called the "components." The variance of this distribution is $(.9)(1) + (.1)(25) = 3.4$ so its standard deviation is 1.844. This distribution is MUCH different that a $N(0, 3.4)$ distribution, even though both distributions have the same mean (0) and variance (3.4). To appreciate this, look at Figure 9.4.

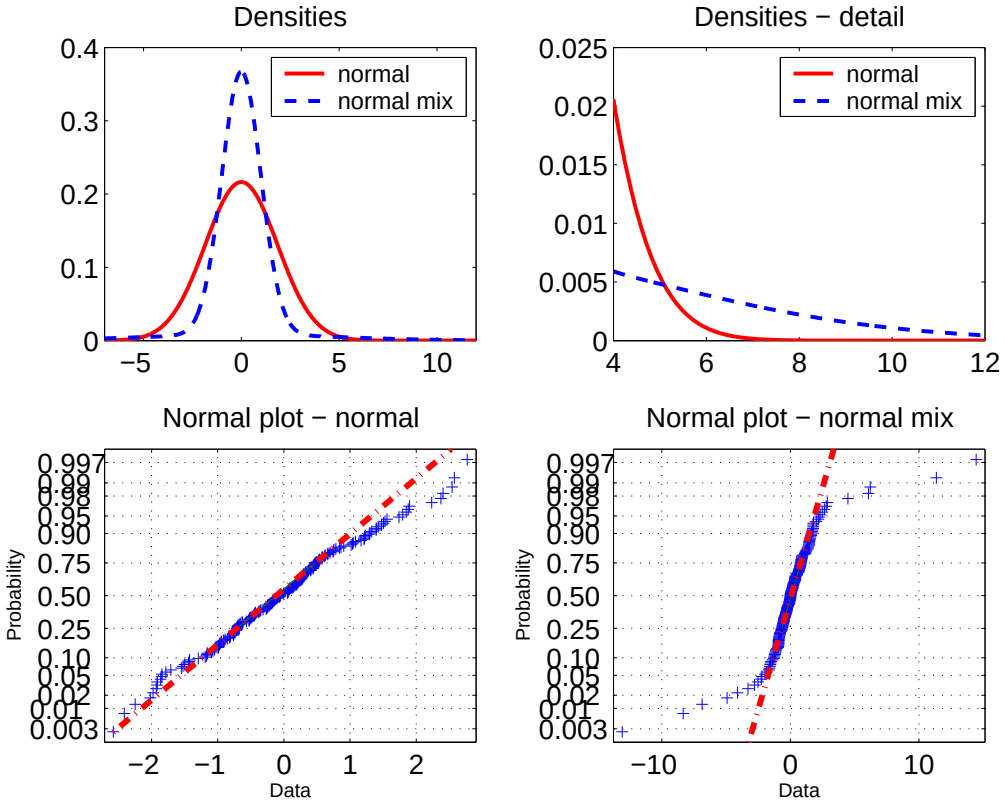

Figure 9.4: *Comparison on normal and heavy-tailed distributions.*

You can see in the top left panel that the two densities look quite different. The normal density looks much more dispersed than the normal mixture, but we know that they actually have the same variances. What's happening? Look at the detail of the right tails in the top right panel. The normal mixture density is much higher than the normal density when x

(the variable on the horizontal axis) is greater than 6. This is the "outlier" region (along with $x < -6$). The normal mixture has more outliers and they come from the 10% of the population with a variance of 25. Remember that $x \pm 6$ is only 6/5 standard deviations from the mean, using the standard deviation 5 of the component they come from. Thus, these observations are not outlying relative to their component standard devation of 5, only relative to the population standard deviation of $\sqrt{3.4}$.

Outliers have a powerful effect on the variance and this small fraction of outliers inflates the variance from 1.0 (the variance of 90% of the population) to 3.4.

Let's see how much more probability the normal mixture distribution has in the outlier range $|x| > 6$ compared to the normal distribution.[3] For a $N(0, \sigma^2)$ random variable X,

$$P\{|X| > x\} = 2(1 - \Phi(x/\sigma)).$$

Therefore, for the normal distribution with variance 3.4,

$$P\{|X| > 6\} = 2\{1 - \Phi(6/\sqrt{3.4})\} = .0011.$$

For the normal mixture population which has variance 1 with probability .9 and variance 25 with probability .1 we have that

$$P\{|X| > 6\} = 2[.9\{1 - \Phi(6)\} + .1\{1 - \Phi(6/5)\}] = (.9)(0) + (.1)(.23) = .023.$$

Since $.023/.001 \approx 21$, the normal mixture distribution is 21 times more likely to be in this outlier range than the $N(0, 3.4)$ population, even though both have a variance of 3.4.

Normal probability plots of samples of size 200 from the normal and the normal mixture distributions are shown in the bottom panels. Notice how the outliers in the normal mixture sample give the data a nonlinear, almost S-shaped, pattern. The deviation of the normal sample from linearity is small and is due entirely to randomness.

In this example, the variance is conditional upon which component of the mixture an observation comes from. The conditional variance is 1 with probability .9 and 25 with probability .1. Because there are only two components, the conditional variance is discrete, in fact, with only two possible values, and the example was easy to analyze. The marginal distribution

[3] There is nothing special about "6" to define the boundary of the outlier range. I just needed a specific number to make numerical comparisons. Clearly, $|x| > 7$ or $|x| > 8$, say, would have been just as appropriate as outlier ranges.

of a GARCH process is also a normal mixture, but with infinitely many components and a continuous distribution of the conditional variance. Although GARCH processes are more complex than the simple model in this section, the same theme applies — conditional heteroscedasticity induces heavy-tailed marginal distributions even though the conditional distributions are normal distributions, which have relatively light tails.

9.8 Comparison of ARMA and GARCH processes

Table 9.8 compares Gaussian white noise, ARMA, GARCH, and ARMA/GARCH processes according to various properties: conditional means, conditional variances, conditional distributions, marginal means, marginal variances, and marginal distributions.

Property	**Gaussian WN**	**ARMA**	**GARCH**	**ARMA/ GARCH**
Cond. mean	constant	non-const	0	non-const
Cond. var	constant	constant	non-const	non-const
Cond. dist'n	normal	normal	normal	normal
Marg. mean & var.	constant	constant	constant	constant
Marg. dist'n	normal	normal	heavy-tailed	heavy-tailed

All of the processes are stationary so that their marginal means and variances are constant. Gaussian white noise is the "baseline" process. Because it is an independent process the conditional distributions are the same as the marginal distribution. Thus, its conditional means and variances are constant and both its conditional and marginal distributions are normal. Gaussian white noise is the "driver" or "source of randomess" behind all the other processes. Therefore, they all have normal conditional distributions just like Gaussian white noise.

9.9 Fitting GARCH models

A time series was simulated using the same program that generated the data in Figure 9.1, the only difference being that 300 observations were generated rather than only 60 as in the figure. The data were saved as "garch02.dat" and analyzed with SAS using the following program.

Listing of the SAS program for the simulated data

```
options linesize = 65 ;
data arch ;
infile 'c:\courses\or473\sas\garch02.dat' ;
input y ;
run ;
title 'Simulated ARCH(1)/AR(1) data' ;
proc autoreg ;
model y =/nlag = 1  archtest garch=(q=1);
run ;
```

This program uses the "autoreg" command that fits AR models. Since nlag = 1, an AR(1) model is being fit. However, the noise is not modeled as independent white noise. Rather an ARCH(1) model is used because of the specification "garch=(q=1)" in the "model" statement below the "autoreg" command. More complex GARCH models can be fit using, for example, "garch=(p=2,q=1)." The specification "archtest" requests tests of ARCH effects, that is, tests the null hypothesis of conditional homoscedasticity versus the alternative of conditional heteroscedasticity.

The output from this SAS program are listed below. The tests of conditional homoscedasticity all reject with p-values of .0001 or smaller. The estimates are $\widehat{\phi} = -.8226$, which is $+.8226$ in our notation. This is close to the true value of 0.8.

The estimates of the ARCH parameters are $\widehat{\alpha}_0 = 1.12$ and $\widehat{\alpha}_1 = .70$. The true values are $\alpha_0 = 1$ and $\alpha_1 = .95$. The standard errors of the ARCH parameters are rather large. This is a general phenomenon; time series usually have less information about variance parameters than about the parameters specifying the conditional expectation. An approximate 95% confidence interval for α_1 is

$$.70 \pm (2)(0.117) = (.446, .934),$$

which does not quite include the true parameter, 0.95. This could have just been bad luck, though it may indicate that $\widehat{\alpha}_1$ is downward biased. The confidence interval is based on the assumption of unbiasedness and is not valid if there is a sizeable bias.

Listing of the SAS output for the simulated data

```
  Simulated ARCH(1)/AR(1) data                  1
   13:01 Wednesday, April 4, 2001

      The AUTOREG Procedure

     Dependent Variable    y

Ordinary Least Squares Estimates

SSE                 2693.22931    DFE                     299
MSE                    9.00746    Root MSE            3.00124
SBC                 1515.48103    AIC                1511.77725
Regress R-Square        0.0000    Total R-Square         0.0000
Durbin-Watson           0.4373

      Q and LM Tests for ARCH Disturbances

Order              Q    Pr > Q            LM    Pr > LM

1         119.7578    <.0001      118.6797     <.0001
2         137.9967    <.0001      129.8491     <.0001
3         140.5454    <.0001      131.4911     <.0001
4         140.6837    <.0001      132.1098     <.0001
5         140.6925    <.0001      132.3810     <.0001
6         140.7476    <.0001      132.7534     <.0001
7         141.0173    <.0001      132.7543     <.0001
8         141.5401    <.0001      132.8874     <.0001
9         142.1243    <.0001      132.8879     <.0001
10         142.6266    <.0001      132.9226     <.0001
11         142.7506    <.0001      133.0153     <.0001
12         142.7508    <.0001      133.0155     <.0001

  Standard               Approx
Variable       DF    Estimate       Error   t Value   Pr > |t|

Intercept       1      0.8910      0.1733      5.14     <.0001

  Estimates of Autocorrelations

 Lag    Covariance     Correlation

   0        8.9774        1.000000
   1        7.0075        0.780567

  Estimates of Autocorrelations

Lag    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

  0    |                   |********************|
  1    |                   |***************     |
```

```
  Simulated ARCH(1)/AR(1) data                  2
   13:01 Wednesday, April 4, 2001

      The AUTOREG Procedure

   Preliminary MSE      3.5076


     Estimates of Autoregressive Parameters

     Standard
  Lag     Coefficient           Error    t Value

    1       -0.780567        0.036209     -21.56


Algorithm converged.


GARCH Estimates

SSE               1056.42037    Observations             300
MSE                  3.52140    Uncond Var        3.72785257
Log Likelihood    -549.43844    Total R-Square        0.6077
SBC               1121.69201    AIC               1106.87688
Normality Test        1.5134    Pr > ChiSq            0.4692


  Standard               Approx
Variable       DF    Estimate       Error   t Value   Pr > |t|

Intercept       1      0.4810      0.3910      1.23     0.2187
AR1             1     -0.8226      0.0266    -30.92     <.0001
ARCH0           1      1.1241      0.1729      6.50     <.0001
ARCH1           1      0.6985      0.1167      5.98     <.0001
```

9.9.1 Example: S&P 500 returns

This example is Example 10.5 in Pindyck and Rubinfeld (1998). The data are monthly from 1960 to 1996. The variables are as follows:

- the S&P 500 index (FSPCOM),
- the return on the S&P 500 (RETURNSP),
- the dividend yield on the S&P 500 index (FSDXP),
- the 3-month T-bill rate (R3),
- the change in the 3-month T-bill rate (DR3),
- the wholesale price index (PW), and

- the rate of wholesale price inflation (GPW).

In this analysis, only RETURNSP, DR3, and GPW are used.

It is expected that variation in stock returns are in part caused by changes in interest rates and changes in the rate of inflation. Therefore, a regression model where RETURNSP is regressed on DR3 and GPW is used. Regression models that regress returns on macroeconomic variables in this way are sometimes called "factor models" — see Bodie, Kane, and Marcus (1999) or Sharpe, Alexander, and Bailey (1999). Figure 9.5 shows the residuals from this regression. The residuals represent the part of the S&P 500 returns that cannot be explained by changes in interest rates and the inflation rate. In the figure, there is some sign of nonconstant volatility. Also, there is no reason to assume that the residuals are uncorrelated as is assumed in a standard regression model. At the very least, this assumption should be checked. If the data contradict the assumption, then a model with correlated errors should be used.

An analysis more appropriate for this data set is to use a regression model to specify the conditional expectation of RETURNSP given DR3 and GPW, but not to assume a "standard" regression model with errors (which the residuals estimate) that are independent white noise. Rather we will assume the model

$$\text{RETURNSP} = \gamma_0 + \gamma_1 \text{DR3} + \gamma_2 \text{GPW} + u_t \tag{9.7}$$

where u_t is an AR(1)/GARCH(1,1) process.[4] Therefore,

$$u_t = \phi_1 u_{t-1} + a_t,$$

where a_t is a GARCH(1,1) process:

$$a_t = \epsilon_t \sigma_t$$

where

$$\sigma_t = \sqrt{\alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2}.$$

Below is a listing of the SAS program used to fit this model. Two models were fit:

[4]We denote the regression coefficients by gamma rather than beta, as is standard, because beta is used for parameters in the GARCH model for a_t.

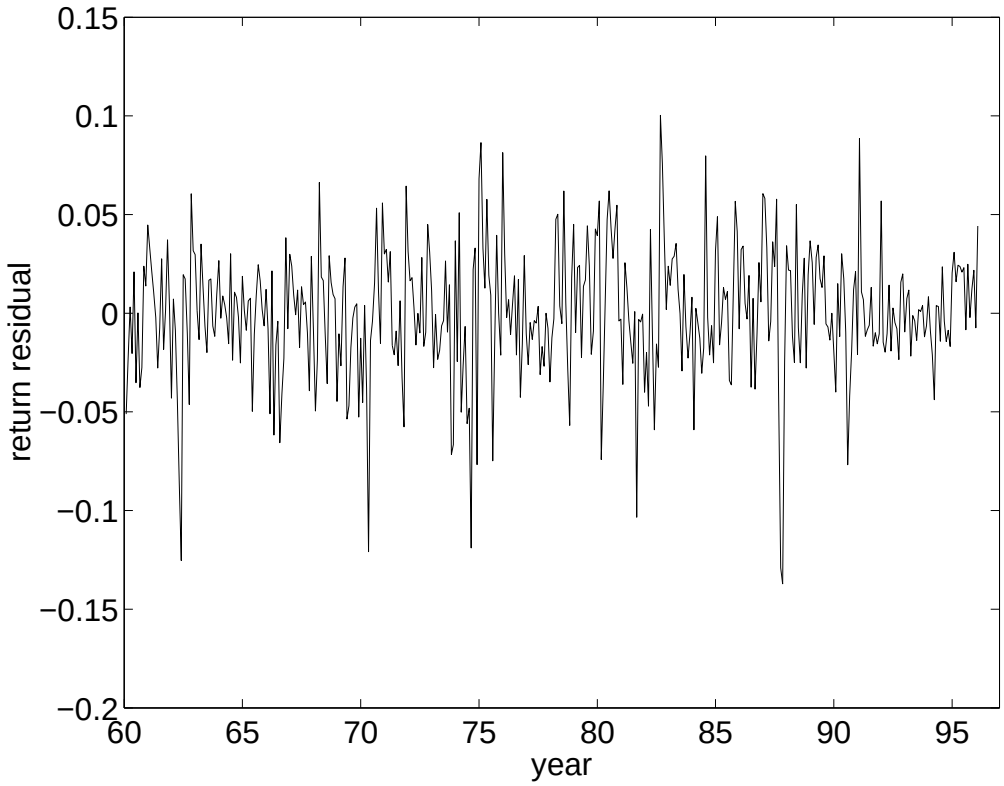

Figure 9.5: *Residuals when the S&P 500 returns are regressed against the change in the 3-month T-bill rates and the rate of inflation.*

- a GARCH(1,1) model with no regression
- the regression model with GARCH(1,1) errors.

The regression model with AR(1)/GARCH(1,1) errors is specified by the command:

```
proc autoreg ;
model returnsp = DR3 gpw/nlag = 1  archtest garch=(p=1,q=1);
```

In this command,

- the statement "returnsp = DR3 gpw " specifies the regression model, that is, that "returnsp" is the dependent variable and "DR3" and "gpw" are the independent variables.
- "nlag = 1" specifies the AR(1) structure.
- "garch=(p=1,q=1)" specifies the GARCH(1,1) structure.
- "archtest" specifies that tests of conditional heteroscedasticity be performed and could have been deleted if we were not interested in testing for GARCH effect, for example, because we already were certain that they existed.

Listing of the SAS program

```
options linesize = 65 ;
data arch ;
infile 'c:\courses\or473\data\pindyck105.dat' ;
input month year RETURNSP FSPCOM FSDXP R3 PW GPW;
DR3 = dif(R3) ;
run ;
title 'S&P 500 monthly data from Pindyck & Rubinfeld, Ex 10.5' ;
title2 'AR(1)/GARCH(1,1) model' ;
proc autoreg ;
model returnsp =/nlag = 1  archtest garch=(p=1,q=1);
run ;
title2 'Regression model with AR(1)/GARCH(1,1)' ;
proc autoreg ;
model returnsp = DR3 gpw/nlag = 1  archtest garch=(p=1,q=1);
run ;
```

The SAS output is listed below.
From examination of the output, the following conclusions can be reached:

- The p-values of the Q and LM tests are all very small, less than .0001. Therefore, the errors in the regression model exhibit conditional heteroscedasticity.
- Ordinary least squares estimates of the regression parameters are:

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
Intercept	1	0.0120	0.001755	6.86	<.0001
DR3	1	-0.8293	0.3061	-2.71	0.0070
GPW	1	-0.8550	0.2349	-3.64	0.0003

- Using residuals from the OLS estimates, the estimated residual autocorrelations are:

Estimates of Autocorrelations

Lag	Covariance	Correlation
0	0.00108	1.000000
1	0.000253	0.234934

- Also, using OLS residuals, the estimate AR parameter is:

Estimates of Autoregressive Parameters

Lag	Coefficient	Standard Error	t Value
1	-0.234934	0.046929	-5.01

- Assuming AR(1)/GARCH(1,1) errors, the estimated parameters of the regression are:

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
Intercept	1	0.0125	0.001875	6.66	<.0001
DR3	1	-1.0665	0.3282	-3.25	0.0012
GPW	1	-0.7239	0.1992	-3.63	0.0003

  - Notice that these differ slightly from OLS estimates.
  - Since all p-values are small, both independent variables are significant.
  - The estimated autoregression parameter is $\widehat{\phi} = 0.2016$.
  - However, the Total R-square value is only 0.0551, so the regression has little predictive value.

- The estimated GARCH parameters are:

AR1	1	-0.2016	0.0603	-3.34	0.0008
ARCH0	1	0.000147	0.0000688	2.14	0.0320
ARCH1	1	0.1337	0.0404	3.31	0.0009
GARCH1	1	0.7254	0.0918	7.91	<.0001

  - the estimate of ϕ is $-.2016$ in SAS's notation but $+.2016$ in our notation. Thus, there is a **positive** association between returns and lagged returns

- Since all p-values are small, all GARCH parameters are significant.

- The GARCH1 estimate (0.7254) is larger than the ARCH1 (0.1337) estimate; this implies that the conditional variance will exhibit reasonably long persistence of volatility.

Listing of SAS output

S&P 500 monthly data from Pindyck & Rubinfeld, Ex 10.5 1
Regression model with AR(1)/GARCH(1,1)
17:04 Tuesday, April 10, 2001

The AUTOREG Procedure

Dependent Variable RETURNSP

Ordinary Least Squares Estimates

SSE	0.46677572	DFE	430
MSE	0.00109	Root MSE	0.03295
SBC	-1711.5219	AIC	-1723.7341
Regress R-Square	0.0551	Total R-Square	0.0551
Durbin-Watson	1.5203		

Q and LM Tests for ARCH Disturbances

Order	Q	Pr > Q	LM	Pr > LM
1	26.8804	<.0001	26.5159	<.0001
2	27.1508	<.0001	27.1519	<.0001
3	28.2188	<.0001	28.4391	<.0001
4	28.6957	<.0001	28.4660	<.0001
5	33.4112	<.0001	32.6168	<.0001
6	34.0892	<.0001	32.6962	<.0001
7	34.4187	<.0001	32.9617	<.0001
8	34.6542	<.0001	32.9636	<.0001
9	35.2228	<.0001	33.3330	0.0001
10	35.3047	0.0001	33.4174	0.0002
11	35.8274	0.0002	33.9440	0.0004
12	36.0142	0.0003	33.9507	0.0007

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
Intercept	1	0.0120	0.001755	6.86	<.0001
DR3	1	-0.8293	0.3061	-2.71	0.0070
GPW	1	-0.8550	0.2349	-3.64	0.0003

Estimates of Autocorrelations

Lag	Covariance	Correlation
0	0.00108	1.000000
1	0.000253	0.234934

```
     S&P 500 monthly data from Pindyck & Rubinfeld, Ex 10.5     2
             Regression model with AR(1)/GARCH(1,1)
                                17:04 Tuesday, April 10, 2001

                       The AUTOREG Procedure

                    Estimates of Autocorrelations

        Lag    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1

          0    |                   |********************|
          1    |                   |*****               |


                     Preliminary MSE     0.00102


                 Estimates of Autoregressive Parameters

                                        Standard
            Lag     Coefficient            Error    t Value

              1       -0.234934        0.046929      -5.01


Algorithm converged.


                            GARCH Estimates

  SSE               0.44176656    Observations             433
  MSE                  0.00102    Uncond Var        0.00104656
  Log Likelihood     889.071523    Total R-Square        0.1058
  SBC                -1735.6479    AIC                -1764.143
  Normality Test        43.0751    Pr > ChiSq            <.0001


                                   Standard                Approx
 Variable       DF    Estimate       Error   t Value   Pr > |t|

 Intercept       1      0.0125    0.001875      6.66     <.0001
 DR3             1     -1.0665      0.3282     -3.25     0.0012
 GPW             1     -0.7239      0.1992     -3.63     0.0003
 AR1             1     -0.2016      0.0603     -3.34     0.0008
 ARCH0           1    0.000147   0.0000688      2.14     0.0320
 ARCH1           1      0.1337      0.0404      3.31     0.0009
 GARCH1          1      0.7254      0.0918      7.91     <.0001
```

9.10 I-GARCH models

I-GARCH or integrated GARCH processes were designed to model data that have persistent changes in volatility. A GARCH(p, q) process is sta-

tionary with a finite variance if

$$\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1.$$

A GARCH(p, q) process is called an I-GARCH process if

$$\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i = 1.$$

I-GARCH processes are either non-stationary or if they are stationary have an infinite variance.

Infinite variance implies heavy-tailed, though a distribution can be heavy-tailed with a finite variance. To appreciate what an infinite variance processes can look like, we will do some simulation. Figure 9.6 shows 40,000 observations of ARCH(1) processes with $\alpha_1 = .9$, 1, and 1.8. The same white noise process is used in each of the ARCH(1) processes. All three ARCH(1) processes are stationary but only the one with $\alpha_1 = .9$ has a finite variance. The second process is an I-GARCH process (actually, I-ARCH since $q = 0$). The third process has $\alpha_1 > 1$ and so is more extreme than an I-GARCH process. Notice how all three processes do revert to their conditional mean of 0. The larger the value of α_1 the more the volatility comes in sharp bursts. The processes with $\alpha_1 = .9$ and $\alpha_1 = 1$ looks similar; there is no sudden change in behavior when the variance becomes infinite. The process with $\alpha_1 = .9$ already has a heavy-tail despite having a finite variance. Increasing α_1 from 0.9 to 1 does not increase the tail weight dramatically.

Normal plots of the simulated data in Figure 9.6 are shown in Figure 9.7. Clearly, the larger the value of α_1, the heavier the tails of the marginal distribution.

None of the processes in Figure 9.6 show much persistence of higher volatility. To model persistence of higher volatility, one needs an I-GARCH(p, q)

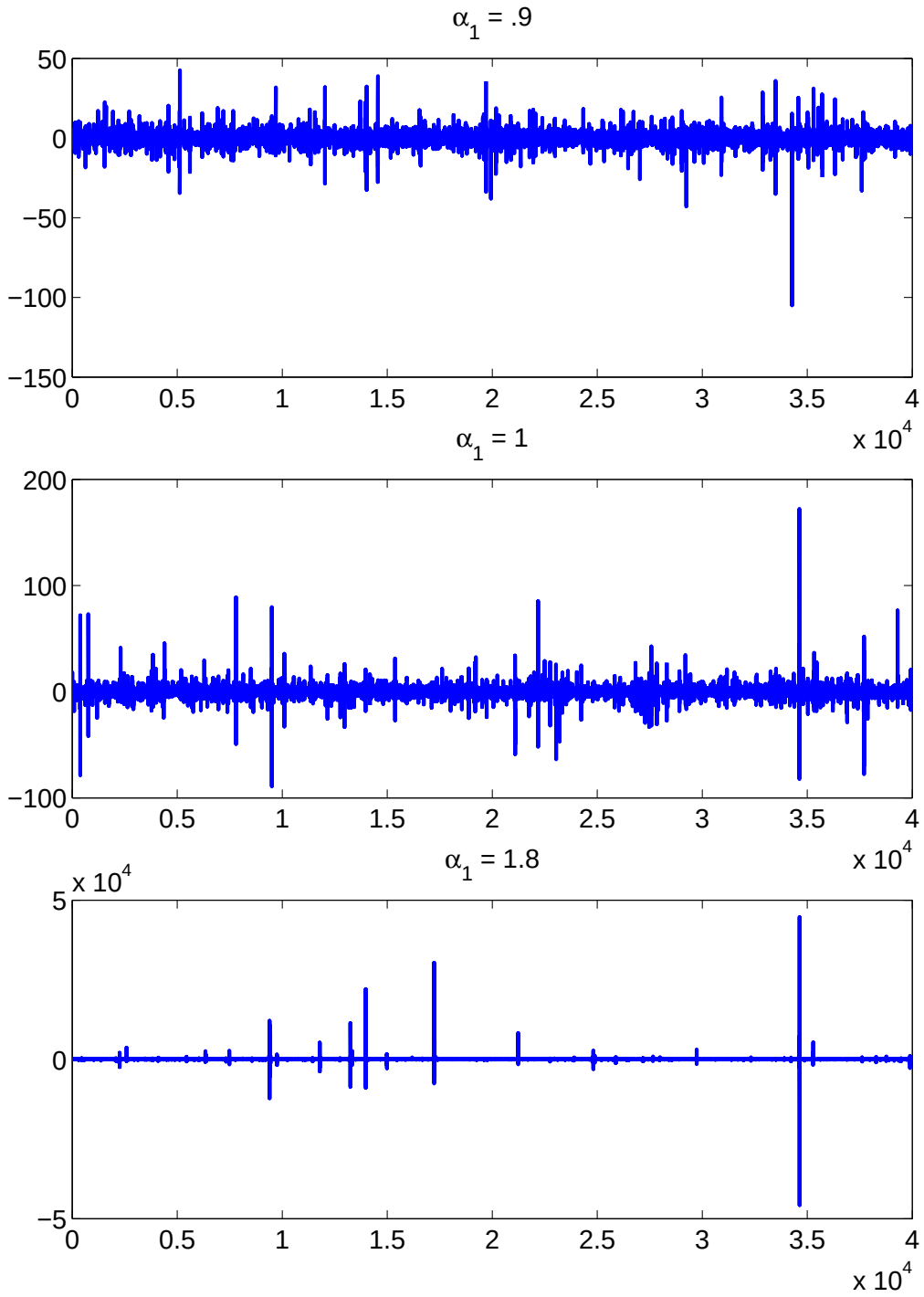

Figure 9.6: *Simulated ARCH(1) processes with $\alpha_1 = .9$, 1, and 1.8. Notice that the vertical scale depends on α_1 because larger values of α_1 result in more extreme outliers.*

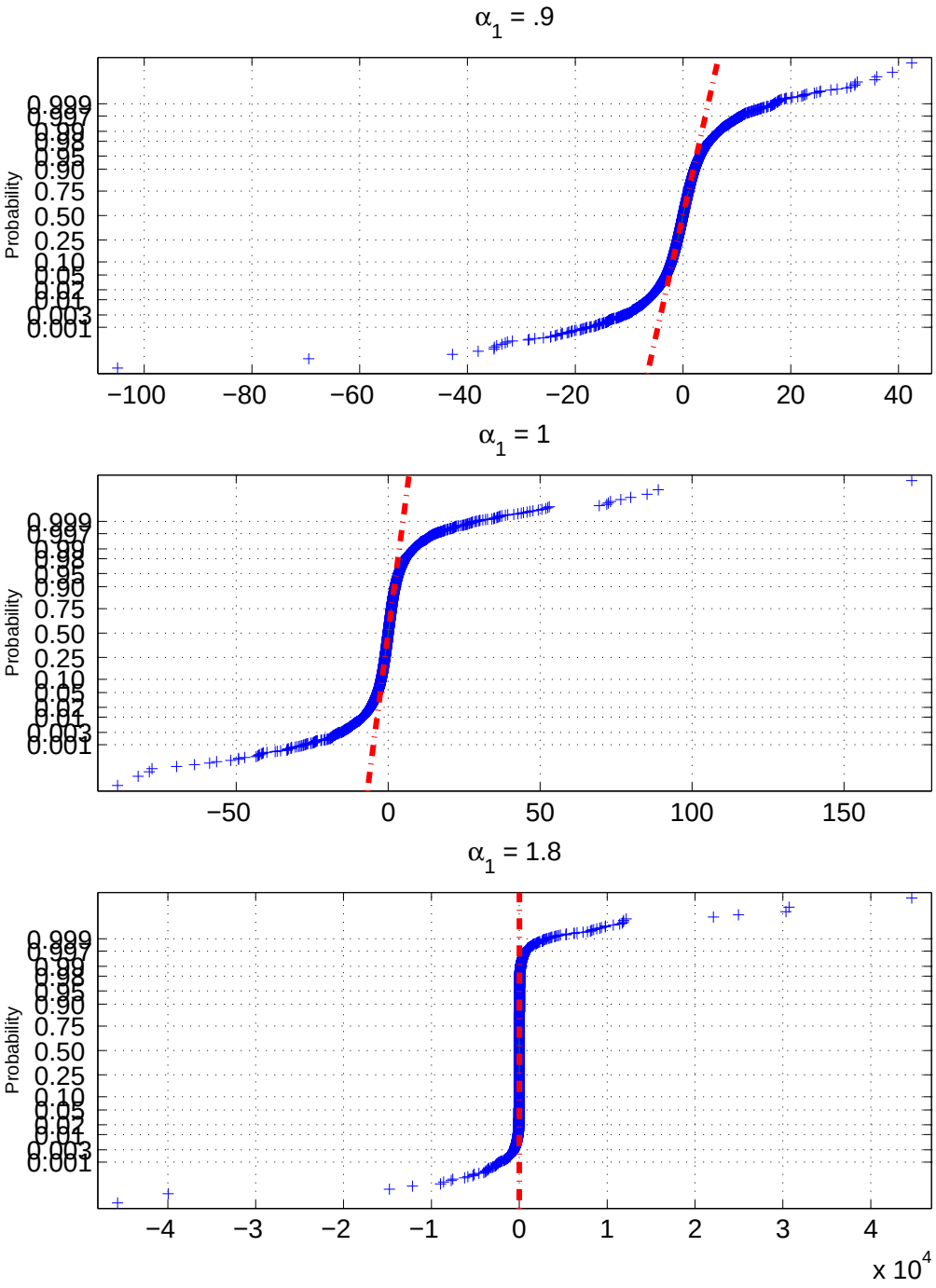

Figure 9.7: *Normal plots of ARCH(1) processes in Figure 9.6.*

process with $q \geq 1$. Figure 9.8 shows simulations from I-GARCH(1,1) processes. Since $\alpha_1 + \beta_1 = 1$ for these processes, $\beta_1 = 1 - \alpha_1$, and the process is completely specified by α_0 and α_1. In this figure, α_0 is fixed at 1 and α_1 is varied. Notice that the conditional variance is very bursty when $\alpha_1 = .95$. When $\alpha_1 = .05$, the conditional standard deviation looks somewhat like a random walk.

I-GARCH processes can be fit by SAS by adding the specification "type = integrated" into the program, e.g., for the previous example with S&P 500 returns:

```
proc autoreg ;
model returnsp =/nlag = 1 garch=(p=1,q=1,type=integrated);
run ;
```

For this example, the I-GARCH(1,1) model seems to fit worse than a GARCH (1,1) model according to AIC; see Section 9.13.

9.10.1 What does it mean to have an infinite variance?

A random variable need not have a finite variance. Also, its expectation need not exist at all. To appreciate these facts, let X be a random variable with density f_X. The expectation of X is

$$\int_{-\infty}^{\infty} x f_X(x) dx$$

provided that this integral is defined. If

$$\int_{-\infty}^{0} x f_X(x) dx = -\infty \tag{9.8}$$

and

$$\int_{0}^{\infty} x f_X(x) dx = \infty \tag{9.9}$$

then the expectation is, formally, $-\infty + \infty$ which is not defined. If integrals on the left hand sides of (9.8) and (9.8) are both finite, then $E(X)$ exists and equals the sum of these two integrals.

Exercise

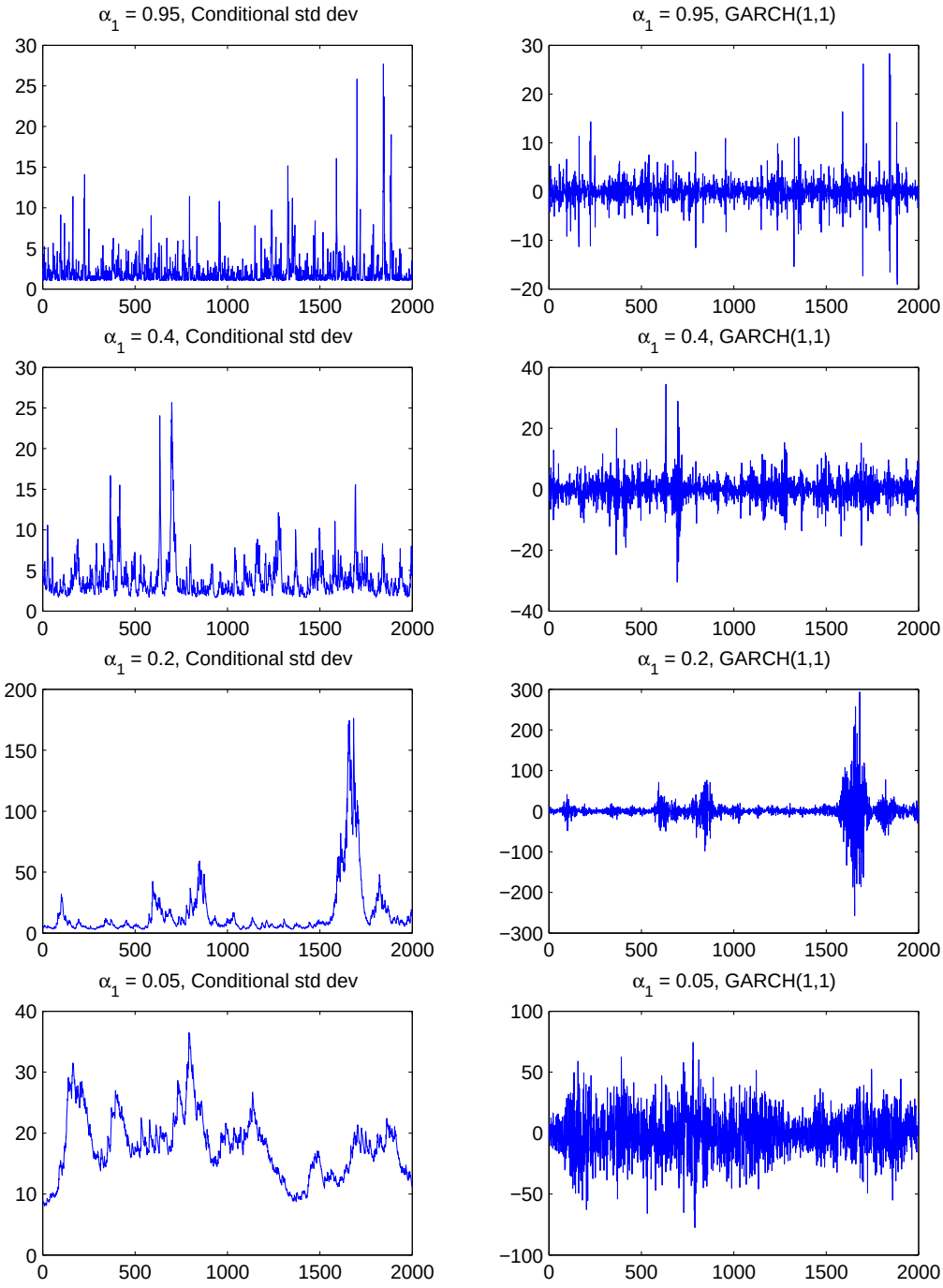

Figure 9.8: *Simulations of I-GARCH(1,1) processes.*

Suppose that $f_X(x) = 1/4$ if $|x| < 1$ and $f_X(x) = 1/(4x^2)$ if $|x| \geq 1$. Show that

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

so that f_X really is a density, but that

$$\int_{-\infty}^{0} xf_X(x)dx = -\infty$$

and

$$\int_{0}^{\infty} xf_X(x)dx = \infty$$

One consequence of the expectation not existing is this. Suppose we have a sample of iid random variables with density f_X. The law of large numbers says that the sample mean will converge to $E(X)$ as the sample size goes to infinity. However, the law of large numbers holds only if $E(X)$ is defined. Otherwise, there is no point to which the sample mean can converge and it will just wander without converging.

Figure 9.9 shows the sample mean of the first t observations plotted against t for the data in Figures 9.6 and 9.10. The sample mean appears to converge to 0 when $\alpha_1 = .9$ or 1, but when $\alpha_1 = 1.8$ it is unclear what the sample mean is doing. The sample mean decays towards 0 when the process is not in a high volatility period, but can shoot up or down during a burst of volatility.

Now suppose that the expectation of X exists and equals μ_X. Then the variance of X equals

$$\int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx.$$

If this integral is $+\infty$, then the variance is infinite.

The law of large numbers also implies that the sample variance will converge to the variance of X as the sample size increases. If the variance of X is infinity, then the sample variance will converge to infinity.

Figure 9.10 shows the sample variance of the first t observations plotted against t for the data in Figure 9.6.

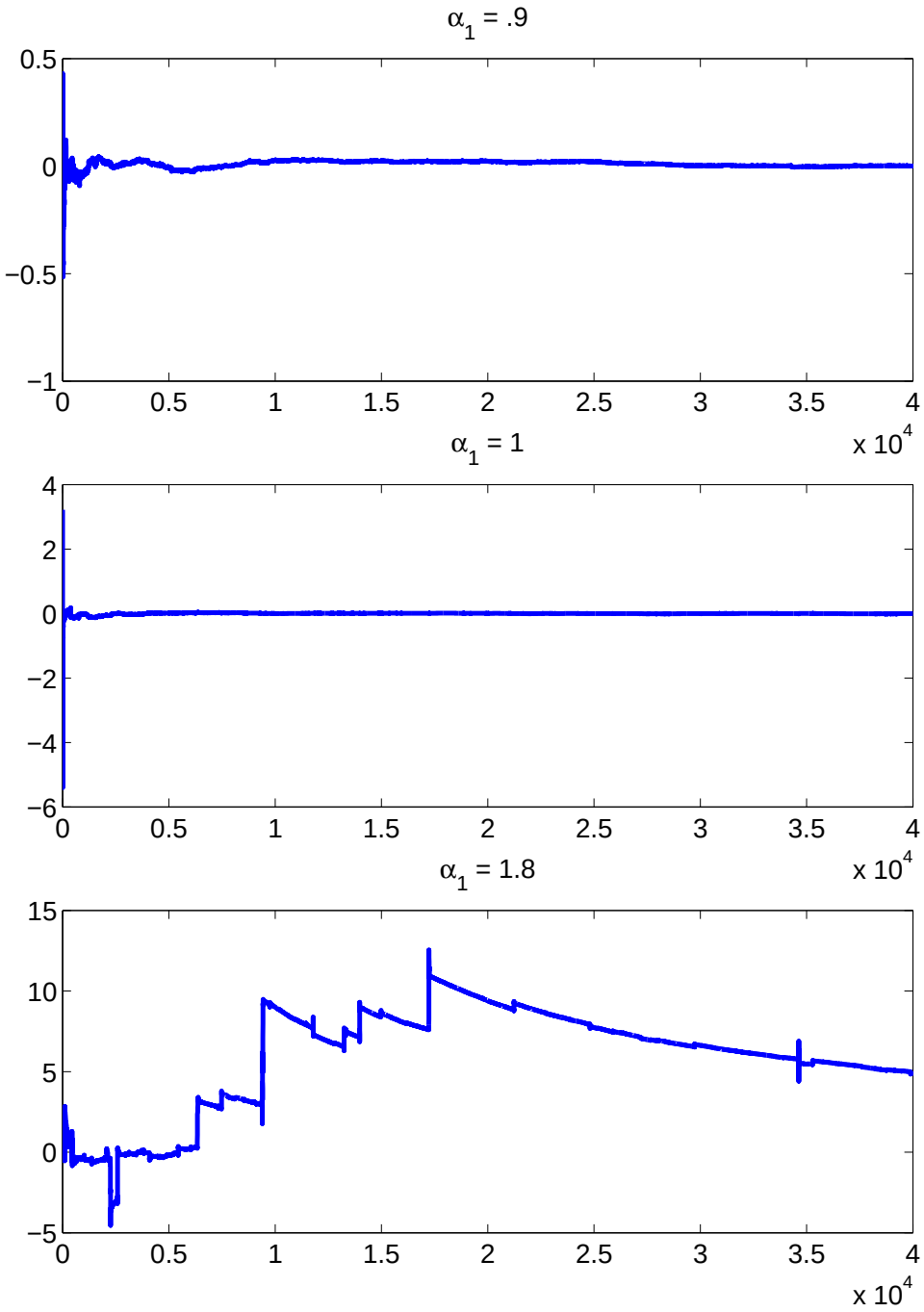

Figure 9.9: *Sample means of simulated ARCH(1) processes with* $\alpha_1 = .9$*, 1, and 1.8.*

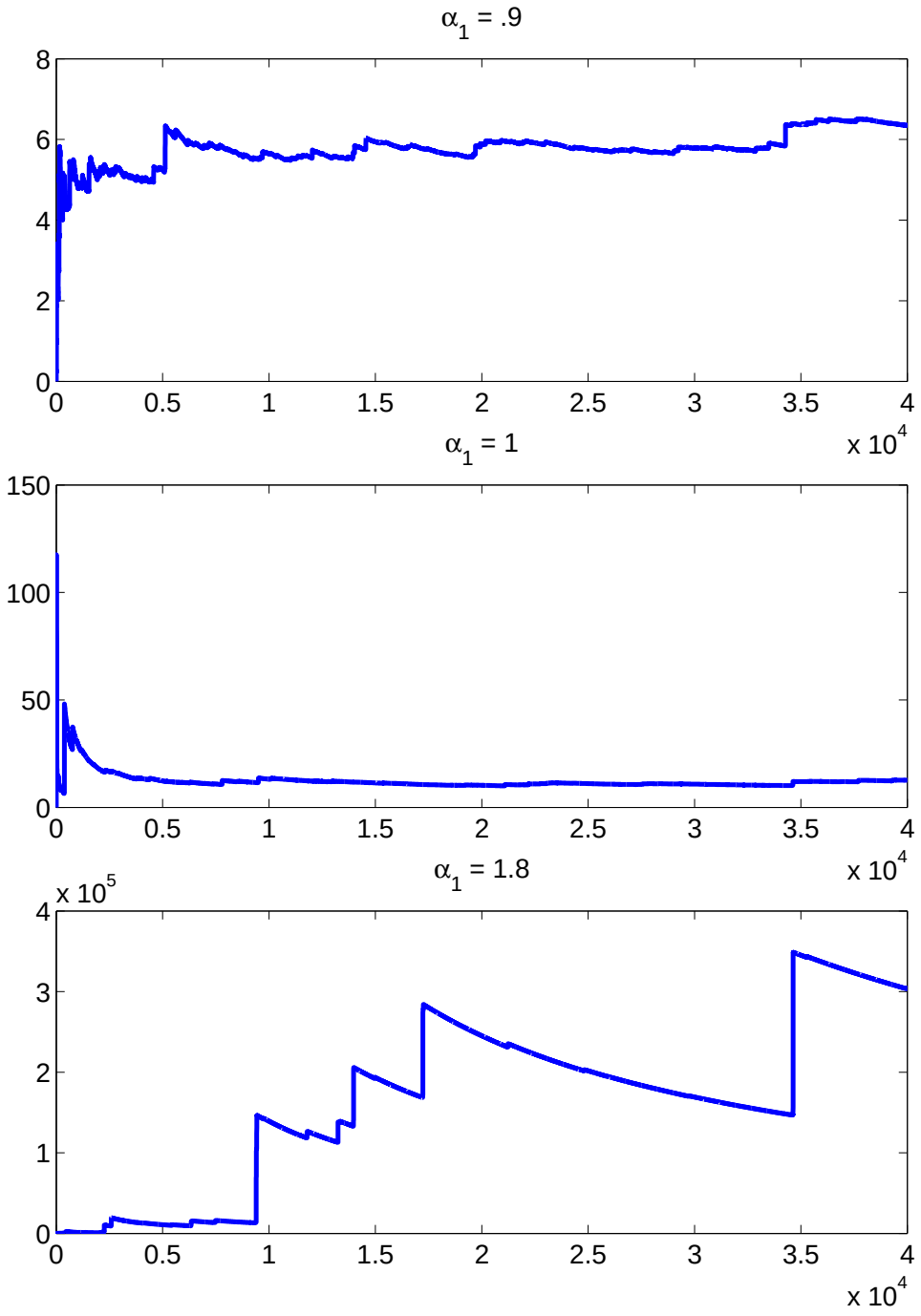

Figure 9.10: *Sample variances of simulated ARCH(1) processes with* $\alpha_1 = .9$*, 1, and 1.8.*

In the top panel, the sample variance should be converging to $10 = (1 - \alpha_1)^{-1}$. Maybe it is converging to 10, but it is hard to tell even with 40,000 observations. In the middle and bottom panels the variance is infinity so the sample variance will converge to infinity. This convergence does appear to be happening in the bottom panel, but it is hard to see in the middle panel. Of course, in the middle panel the value of α_1 is on the borderline between finite and infinite variance, and the infinite variance may take a very long time to have its effect.

9.11 GARCH-M processes

We have seen that one can fit regression models with AR/GARCH errors. In fact, we have done that with the S&P 500 data. In some examples, it makes sense to use the conditional standard deviation as one of the regression variables. For example, when the dependent variable is a return we might expect that higher conditional variability will cause higher returns. This is because the market demands a higher risk premium for higher risk.

Models where the conditional standard deviation is a regression variable are called GARCH-in-mean, or GARCH-M, models. They have the form

$$Y_t = \boldsymbol{X}_t^{\mathsf{T}}\boldsymbol{\gamma} + \delta\sigma_t + a_t,$$

where a_t is a GARCH process with conditional standard deviation σ_t. In this model, σ_t and the components of $\boldsymbol{X}_t$ are the predictor variables and δ and the components of γ are the regression coefficients.

GARCH-M models can be fit in SAS by adding the keyword "mean" to the GARCH specification, e.g.,

```
proc autoreg ;
model returnsp =/nlag = 1 garch=(p=1,q=1,mean);
run ;
```

or for I-GARCH-M

```
proc autoreg ;
model returnsp =/nlag = 1 garch=(p=1,q=1,mean,type=integrated);
run ;
```

For the S&P 500 returns data, a GARCH(1,1)-M was fit in SAS. The estimate of $\widehat{\delta}$ was .5150 with a standard error of .3695. This gives a t-value of 1.39 and a p-value of .1633. Since the p-value is reasonably large we could accept the null hypothesis that $\delta = 0$. Therefore, we see no strong evidence that there are higher returns during times of higher volatility. The volatility of the S&P 500 is *market risk* so this finding is a bit surprising. It may be that the effect is small ($\widehat{\delta}$ is positive, after all) and cannot be detected with certainty. The AIC criterion *does* select the GARCH-M model; see Section 9.13.

9.12 E-GARCH

In finance, the "leverage effect" predicts that an asset's price will become more volatile when its price decreases. E-GARCH processes were designed, inter alia, to model the leverage effect.

The exponential GARCH, or E-GARCH, model is

$$\log(\sigma_t) = \alpha_0 + \sum_{i=1}^{q} \alpha_1 g(\epsilon_{t-i}) + \sum_{i=1}^{p} \beta_i \log(\sigma_{t-i}),$$

where

$$g(\epsilon_t) = \theta\epsilon_t + \gamma\{|\epsilon_t| - E(|\epsilon_t|)\}$$

and $\epsilon_t = a_t/\sigma_t$. Since $\log(\sigma_t)$ can be negative, there are no constraints on the parameters, which makes computations easier since constrained optimization is not needed when computing the maximum likelihood estimate.

Notice that

$$g(\epsilon_t) = -\gamma E(|\epsilon_t|) + (\gamma + \theta)|\epsilon_t| \quad \text{if} \quad \epsilon_t > 0,$$

and

$$g(\epsilon_t) = -\gamma E(|\epsilon_t|) + (\gamma - \theta)|\epsilon_t| \quad \text{if} \quad \epsilon_t < 0,$$

Exercise: Show that $E(|\epsilon_t|) = \sqrt{2/\pi}$ = .7979.

Typically, $-1 < \widehat{\theta} < 0$ so that $0 < \gamma + \theta < \gamma - \theta$. For example, $\widehat{\theta} = -.7$ in the S&P 500 example; see below. The function g with $\theta = -.7$ is plotted in the top left panel of Figure 9.11. Notice that $g(\epsilon_t)$ is negative if $|\epsilon_t|$ is close to zero; small values of noise decrease σ_t. If $|\epsilon_t|$ is large, then σ_t increases. With a negative value of θ, σ_t increases more rapidly as a function of $|\epsilon_t|$ when ϵ_t is negative than when ϵ_t is positive,

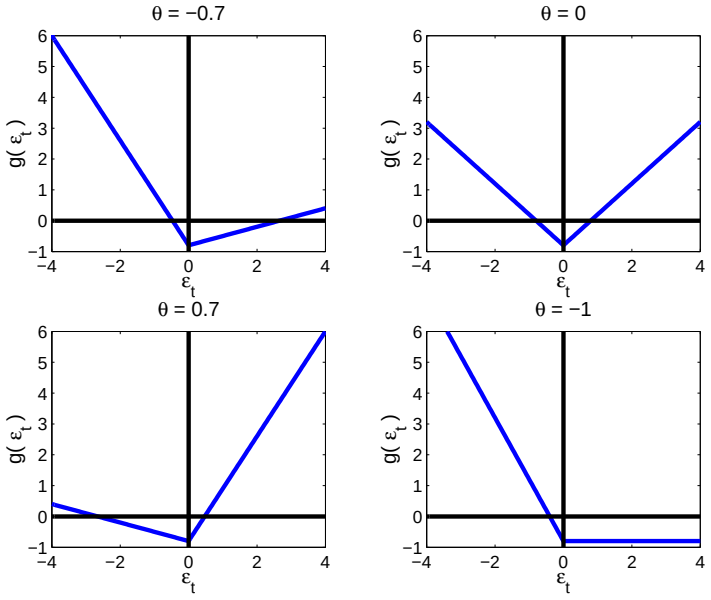

Figure 9.11: *The g function for the S&P 500 data (top left panel) and several other values of θ.*

As mentioned previously, the "leverage effect" occurs when an asset's price becomes more volatile as its price decreases. This is the type of behavior obtained when $\theta < 0$. The ability to accommodate leverage effects was the reason that the E-GARCH model was introduced by 'Daniel Nelson.

The function g for several other values of θ are also shown in Figure 9.11. When $\theta = 0$ (top right) the function is symmetric about 0. The bottom right panel where $\theta = -1$ shows an extreme case where $g(\epsilon_t)$ is negative for all positive ϵ_t.

SAS fits the E-GARCH model with γ fixed as 1 and θ estimated.[5] The E-GARCH model is specified by using "type=exp" as in

```
proc autoreg ;
model returnsp =/nlag = 1 garch=(p=1,q=1,mean,type=exp);
run ;
```

[5]The data can only determine the ratio γ/θ, so one of γ or θ must be fixed at an arbitrary value. SAS uses $\gamma = 1$.

This command specifies both a GARCH-in-mean effect and the E-GARCH model. Omitting "mean" removes the GARCH-in-mean effect.

9.13 Back to the S&P 500 example

SAS can fit six different AR(1)/GARCH(1,1) models since SAS allows "type" to be "integrated," "exp," or "nonneg." The last is the default and specifies a GARCH model with non-negativity constraints. Moreover, for each of these three types we can specify that a GARCH-in-mean effect be included or not. Table 9.1 contains the AIC statistics for the six models. The models are ordered from best fitting to worse fitting according to AIC—remember that a smaller AIC is better.

Model	**AIC**	**Δ AIC**
E-GARCH-M	-1783.9	0
E-GARCH	-1783.1	0.8
GARCH-M	-1764.6	19.3
GARCH	-1764.1	19.8
I-GARCH-M	-1758.0	25.9
I-GARCH	-1756.4	27.5

Table 9.1: AIC statistics for six AR(1)/GARCH(1,1) models fit to the S&P 500 returns data. Δ AIC is the change in AIC between a given model and E-GARCH-M.

It seems that the E-GARCH-M model is best, though the E-GARCH model fits nearly as well. The E-GARCH-M model will be used in the remaining discussion. To see if more AR or GARCH parameters would improve the fit, AR(2) and E-GARCH(1,2)-M, E-GARCH(2,1)-M, and E-GARCH(2,2)-M models were tried, but none of these lowered AIC or had all parameters significant at $p = .1$. Thus, AR(1)/E-GARCH(1,1) appears to be a good fit to the noise and adding a GARCH-in-mean term to the regression model seems reasonable although it does not improve the fit very much.

The fit to this model is in the SAS output listed below.

Listing of SAS output for the E-GARCH-M model:

```
      S&P 500 monthly data from Pindyck & Rubinfeld, Ex 10.5     2
           Regression model with AR(1)/E-GARCH(1,1)-M
                                11:52 Sunday, April 15, 2001

                     The AUTOREG Procedure

              Estimates of Autoregressive Parameters

                                Standard
          Lag     Coefficient      Error    t Value

            1       -0.234934    0.046929      -5.01

Algorithm converged.

                  Exponential GARCH Estimates

  SSE               0.44211939    Observations             433
  MSE                  0.00102    Uncond Var                 .
  Log Likelihood     900.962569    Total R-Square        0.1050
  SBC               -1747.2885    AIC                -1783.9251
  Normality Test       24.9607    Pr > ChiSq             <.0001

                                Standard                 Approx
 Variable       DF    Estimate      Error   t Value   Pr > |t|

 Intercept       1   -0.003791     0.0102     -0.37     0.7095
 DR3             1     -1.2062     0.3044     -3.96     <.0001
 GPW             1     -0.6456     0.2153     -3.00     0.0027
 AR1             1     -0.2376     0.0592     -4.01     <.0001
 EARCH0          1     -1.2400     0.4251     -2.92     0.0035
 EARCH1          1      0.2520     0.0691      3.65     0.0003
 EGARCH1         1      0.8220     0.0606     13.55     <.0001
 THETA           1     -0.6940     0.2646     -2.62     0.0087
 DELTA           1      0.5067     0.3511      1.44     0.1490
```

9.14 The GARCH zoo

There are many more types of GARCH models than the few mentioned so far. I've discussed only the most widely used models that can be fit in SAS. The number of models seems limited only by the number of letters in

the alphabet, not the imagination of econometricians! Here's a sample of other GARCH models mentioned in Bollerslev, Engle, and Nelson (1994) and elsewhere:

- A-GARCH = asymmetric GARCH
- M-GARCH = multivariate GARCH
- QARCH = quadratic ARCH
- TARCH = threshold ARCH
- STARCH = structural ARCH
- SWARCH = switching ARCH
- QTARCH = quantitative threshold ARCH
- vector ARCH
- diagonal ARCH
- factor ARCH

9.15 Applications of GARCH in finance

GARCH models were developed by econometricians working with business and finance data, and their applications to finance have been extensive. The review paper by Bollerslev, Engle, and Nelson lists hundreds of references.

Finance models such as the CAPM and the Black-Scholes model for option pricing assume a constant conditional variance. When this assumption is false, use of these models can lead to serious errors. Therefore, generalization of finance models to include GARCH errors has been a hot topic. See Bollerslev, Engle, and Woolridge (1988) and Duan (1996a, 1996b) for some examples of finance models with conditional heteroscedasticity.

Rossi (1996) is a collection of papers, many reprinted from finance journals, on modeling stock market volatility with GARCH models.

9.16 Summary

- The marginal, or unconditional, distribution of a stationary process is the distribution of an observation from the process given no information about the previous or future observations
  - by stationarity the marginal distribution must be constant
  - in particular, the marginal mean and variance are constant.
- Besides the marginal distribution, we are interested in the conditional distribution of the next observation given the current information set of present and past values of the process, and perhaps of other processes.
- For ARMA processes the conditional mean is non-constant but the conditional variance is constant.
- The constant conditional variance of ARMA processes makes them unsuitable for modeling the volatility of financial markets.
- GARCH process have non-constant conditional variance and were developed to model volatility.
- GARCH processes can be used as the "noise" term of an ARMA process
  - ARMA/GARCH processes have both non-constant conditional mean and a non-constant conditional variance.
  - GARCH and ARMA/GARCH processes can be estimated by maximum likelihood.
  - PROC AUTOREG in SAS fits AR/GARCH models
- The simple ARCH(q) models have burst of volatility but cannot model persistent volatility.
- The generalized ARCH (GARCH) models can model persistent volatility.
- The marginal distribution of a GARCH process has heavier tails than the normal distribution.

  - heavy tails = outlier prone
  - in fact, for certain parameter values a GARCH process will have an infinite variance, which is an extreme case of heavy tails
    * I-GARCH (integrated GARCH) models are examples of GARCH models with infinite variance.

- If the marginal variance is infinite, then the sample variance will converge to infinity as the sample size increase
- For extremely heavy tails, the marginal expectation may not exist
  - then there exists no point to which the sample mean can converge
    * the sample mean will wander aimlessly.
- ARMA/GARCH processes can be used as the noise term in regression models
  - SAS's PROC AUTOREG can use an AR/GARCH noise term in a regression. model
- The **GARCH-M** models use the conditional standard deviation as an independent variable in the regression.
- The "leverage effect" occurs when a negative return (drop in price) increases the volatility of future returns because the denominator of those returns is smaller.
- E-GARCH models were designed to capture the leverage effect
  - in an E-GARCH model, the log of the conditional standard deviation is modeled as an ARMA process but with the white noise process ϵ_t replaced by another white noise process $g(\epsilon_t)$
  - there is no need for non-negativity constraints on the parameters, such as those in an ordinary GARCH model, since the log standard deviation can be negative
  - the parameter θ in an E-GARCH model determines the leverage effects

    - $\theta < 0 \Rightarrow$ leverage effects
    - $\theta = 0 \Rightarrow$ no leverage
    - $\theta > 0 \Rightarrow$ positive returns increase volatiltiy (this would be the opposite of the leverage effect and is not expected to happen in practice).

- In the S&P 500 example we found that
    - returns are negatively associated with changes in interest rates (an increase in interest rates decreases returns)
    - returns are negatively associated with changes in wholesale prices
    - returns are positively associated with returns lagged one month ($\widehat{\phi} = -.2376$ is negative in the SAS output, but SAS's definition of ϕ is the negative of ours—our $\widehat{\phi}$ is $+.2376$)
    - there are leverage effects since a E-GARCH model fits better than a GARCH model and $\widehat{\delta} = -.7$
    - there is slight evidence of a GARCH-in-mean effect, that is, there is some reason to believe that there is a risk premium.

- There is a wide variety of other GARCH models in the literature, but the ones discussed here, ARCH(q), GARCH(p, q), E-GARCH, GARCH-M, and I-GARCH, are probably enough to know about since they can model a wide variety of data types
    - the models discussed in these notes are the ones that can be fit by SAS.

- There is a large and growing literature on financial models with returns following GARCH processes.

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Chapter 10

Fixed Income Securities: 5/21/02

10.1 Introduction

Corporations finance their operations by selling stock and bonds. Owning a share of stock means partial ownership of the company. You share in both the profits and losses of the company, so nothing is guaranteed.

Owning a bond is different. When you buy a bond you are loaning money to the corporation. The corporation is obligated to pay back the principle and to pay interest as stipulated by the bond. You receive a fixed stream of income, unless the corporation defaults on the bond. For this reason, bonds are called "fixed-income" securities.

It might appear that bonds are risk-free, almost stodgy. This is not the case. Many bonds are long-term, e.g., 20 or 30 years. Even if the corporation stays solvent or if you buy a US Treasury bond where default is for all intents and purposes impossible, your income from the bond is guaranteed only if you keep the bond to maturity. If you sell the bond before maturity, your return will depend on changes in the price of the bond. Bond prices change due to changes in interest rates.

The interest rate on your bond is fixed, but in the market interest rates fluctuate. Therefore, the market value of your bond fluctuations too. For example, if you buy a bond paying 5% and the rate of interest increases to 6% then your bond is inferior to new bonds offering 6%. Consequently,

the price of your bond will decrease. If you sell the bond you would lose money. So much for a "fixed income" stream!

If you ever bought a CD, which really is a bond that you buy from a bank or credit union, you will have noticed that the interest rate depends on the maturity of the CD. This is a general phenomenon. For example, on March 28, 2001, the interest rate of Treasury bills[1] was 4.23% for 3-month bills. The yields on Treasurys were 4.41%, 5.01%, and 5.46% for 2, 10, and 30 year maturities, respectively. The **term structure** of interest rates describes how rates of interest change with the maturity of bonds.

In this chapter we will study how bond prices fluctuate due to interest rate changes. We will also study how the term structure of interest rates can be determined.

10.2 Zero coupon bonds

Zero-coupon bonds, also called pure discount bonds, pay no principle or interest until maturity. A "zero" has a par value which is the payment made to the bond holder at maturity. The zero sells for less than par, which is the reason it is a "discount bond."

For example, consider a 20-year zero with a par value of \$1000 and 6% interest compounded annually. The price is the present-value of \$1000 with discounting annually at 6%. That is, the price is

$$\frac{\$1000}{(1.06)^{20}} = \$311.80.$$

If the interest is 6% but compounded every six months, then the price is

$$\frac{\$1000}{(1.03)^{40}} = \$306.56,$$

and if the interest is 6% compounded continuously then the price is

$$\frac{\$1000}{\exp\{(.06)(20)\}} = \$301.19.$$

[1] Treasury bills have maturities of one year or less, Treasury notes have maturities from one to ten years, and Treasury bonds have maturities from 10 to 30 years.

10.2.1 Price and returns fluctuate with the interest rate

For concreteness, assume semi-annual compounding. Suppose you just bought the zero for \$306.56 and then six months later the interest rate increased to 7%. The price would now be

$$\frac{\$1000}{(1.035)^{39}} = \$261.41$$

so your investment would drop by (\$306.56 − \$261.41) = \$45.15. You will still get your \$1000 if you keep the bond for 20 years, but if you sell it now you will lose \$45.15. This is a return of

$$\frac{-45.15}{306.56} = -14.73\%$$

for a half-year or -29.46% per year. And the interest rate only changed from 6% to 7%!

If the interest rate dropped to 5% after six months, then your bond would be worth

$$\frac{\$1000}{(1.025)^{39}} = \$381.74.$$

This would be an annual rate of return of

$$2\left(\frac{381.74 - 306.56}{306.56}\right) = 49.05\%.$$

If the interest rate remained unchanged at 6%, then the price of the bond would be

$$\frac{\$1000}{(1.03)^{39}} = \$315.75.$$

The annual rate of return would be

$$2\left(\frac{315.75 - 306.56}{306.56}\right) = 6\%.$$

Thus, if the interest rate does not change, you can earn a 6% annual rate of return by selling the bond before maturity. If the interest rate does change, however, the 6% annual rate of return is guaranteed only if you keep the bond until maturity.

General formula

The price of a zero coupon bond is given by

$$\text{PRICE} = \text{PAR}(1+r)^{-T}$$

if T is the time to maturity in years and the annual rate of interest is r with annual compounding. If we assume the interest rate is r per half-year with semi-annual compounding, then the price is

$$\text{PRICE} = \text{PAR}(1+r)^{-2T} \tag{10.1}$$

10.3 Coupon bonds

Coupon bonds make regular interest payments.[2] Coupon bonds generally sell at par when issued. At maturity, one receives the principle and the final interest payment.

As an example, consider a 20-year coupon bond with a par value of \$1000 and 6% annual interest with semi-annual coupon payments. Each coupon payment will be \$30. Thus, the bond holder receives 40 payments of \$30, one every six months plus a principle payment of \$1000 after 20 years. One can check that the present value of all payments, with discounting at the 6% annual rate (3% semi-annual), equals \$1000:

$$\sum_{t=1}^{40} \frac{30}{(1.03)^t} + \frac{1000}{(1.03)^{40}} = 1000.$$

After six months if the interest rate is unchanged, then the bond (including the first coupon payment which is now due) is worth

$$\sum_{t=0}^{39} \frac{30}{(1.03)^t} + \frac{1000}{(1.03)^{39}} = (1.03)\left(\sum_{t=1}^{40} \frac{30}{(1.03)^t} + \frac{1000}{(1.03)^{40}}\right) = 1030,$$

which is a 6% annual return as expected. If the interest rate increases to 7%, then after six months the bond (plus the interest due) is only worth

$$\sum_{t=0}^{39} \frac{30}{(1.035)^t} + \frac{1000}{(1.035)^{39}} = (1.035)\left(\sum_{t=1}^{40} \frac{30}{(1.035)^t} + \frac{1000}{(1.035)^{40}}\right) = 924.49.$$

[2] At one time actual coupons were attached to the bond, one coupon for each interest payment. When a payment was due, its coupon could be clipped off and sent to the issuing company for payment.

This is an annual return of

$$2\left(\frac{924.49-1000}{1000}\right) = -15.1\%.$$

If the interest rate drops to 5% after six months then the investment is worth

$$\sum_{t=0}^{39}\frac{30}{(1.025)^t} + \frac{1000}{(1.025)^{39}} = (1.025)\left(\sum_{t=1}^{40}\frac{30}{(1.025)^t} + \frac{1000}{(1.025)^{40}}\right) = 1,153.70, \tag{10.2}$$

and the annual return is

$$2\left(\frac{1153.6-1000}{1000}\right) = 30.72\%.$$

Some general formulas

Let's derive some useful formulas. If a bond with a par value of PAR matures in T years and makes semi-annual payments of C and the discount rate (rate of interest) is r per half-year, then the value of the bond when it is issued is

$$\begin{aligned}\sum_{t=1}^{2T}\frac{C}{(1+r)^t} \quad + \quad \frac{\text{PAR}}{(1+r)^{2T}} &= \frac{C}{r}\left\{1-(1+r)^{-2T}\right\} + \frac{\text{PAR}}{(1+r)^{2T}} \\ &= \frac{C}{r} + \left\{\text{PAR} - \frac{C}{r}\right\}(1+r)^{-2T}. \end{aligned} \tag{10.3}$$

This formula will be derived soon.

If $C = \text{PAR}\times r$, then the value of the bond when issued is PAR. The value six months later is $(1+r)$ times the value in equation (10.3). The MATLAB function "bondvalue.m" computes (10.3). The call to this function is `bondvalue(c,T,r,par)`.

For example, if the coupon is $C = 30$, if $T = 30$, and if after six months $r = 6.2\%$/half-year (or 3.1%/year), then the bond is worth

$$\frac{30}{.031} + \left\{1000 - \frac{30}{.031}\right\}(1.031)^{-59} = 1003.1.$$

This value was computed by MATLAB with the call

```
bondvalue(30,29.5,.031,1000).
```

Similarly, (10.2) was computed with the MATLAB call

```
1.025*bondvalue(30,20,.025,1000).
```

Derivation of (10.3)

The summation formula for a finite geometric series is

$$\sum_{i=0}^{T} r^i = \frac{1 - r^{T+1}}{1 - r}, \tag{10.4}$$

provided that $r \neq 1$.

Exercise: Derive (10.4).

Therefore,

$$\sum_{t=1}^{2T} \frac{C}{(1+r)^t} = \frac{C}{1+r} \sum_{t=0}^{2T-1} \frac{1}{(1+r)^t} = \frac{C\{1-(1+r)^{-2T}\}}{(1+r)(1-(1+r)^{-1})} \tag{10.5}$$

$$= \frac{C}{r}\{1-(1+r)^{-2T}\}. \tag{10.6}$$

10.4 Yield to maturity

Suppose a bond with $T = 30$ and $C = 40$ is selling for \$1200, \$200 above par. If the bond were selling at par, then the interest rate would be .04/half-year (= .08/year). The 4%/half-year rate is called the **coupon rate**.

But the bond is *not* selling at par. If you purchase the bond at \$1200 you will make *less* than 8% per year interest. There are two reasons that the rate of interest is less than 8%. First, the coupon payments are \$40 or 40/1200 = 3.333%/half-year (or 6.67%/year) for the \$1200 investment; 6.67%/year is called the **current yield**. Second, at maturity you only get back \$1000, not the entire \$1200 investment. The current yield of 6.67%/year, even though less than the coupon rate of 8%/year, overestimates the return since it does not account for this loss of capital.

The **yield to maturity** is a measure of the average rate of return, including the loss (or gain) of capital because the bond was purchased above (or below) par. For this bond, the yield to maturity is the value of r that solves

$$1200 = \frac{40}{r} + \left\{1000 - \frac{40}{r}\right\}(1+r)^{-60}. \tag{10.7}$$

The right hand side of (10.7) is (10.6) with $C = 40$, $T = 30$, and PAR $= 1000$. It is easy to solve equation (10.7) numerically. The MATLAB program yield.m does the following:

- computes the bond price for each r value on a grid
- graphs bond price versus r (this is not necessary but it's fun to see the graph)
- interpolates to find the value of r when bond value equals 1200

One finds that the yield to maturity is 0.0324, that is, 3.24%/half-year. Figure 10.1 shows the graph of bond price versus r and shows that $r = .0324$ maps to a bond price of $1200.

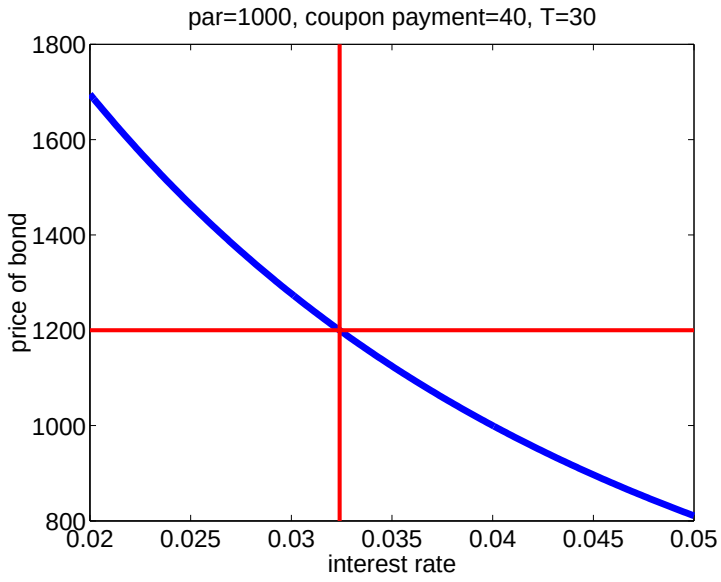

Figure 10.1: *Bond price versus the interest rate r and determining by interpolation the yield to maturity when the price equals $1200.*

The yield to maturity of .0324 is less than the current yield of 0.0333 which is less than the coupon rate of 40/1000 = .04. (All three rates are rates per half-year.) Thus, we see that

- coupon rate > current yield
  - since the bond sells above par

- current yield $>$ yield to maturity
  - since yield to maturity accounts for the loss of capital when at the maturity date you only get back \$1000 of the \$1200 investment.

Whenever, as in this example, the bond is selling above par, we have

$$\text{coupon rate} > \text{current yield} > \text{yield to maturity.} \tag{10.8}$$

Everything is reversed if the bond is selling below par. For example, if the price of the bond were only \$900, then

- the yield to maturity would be 0.0448 (as before, this value can be determined by "yield.m" using interpolation)
- the current yield would be $40/900 = 0.0444$
- the coupon rate would still be $40/1000 = .04$.

Therefore, when the bond is selling below par we would have

$$\text{coupon rate} < \text{current yield} < \text{yield to maturity,} \tag{10.9}$$

which is just the opposite of (10.8) which applies to bonds selling above par.

General method for yield to maturity

The yield to maturity (on an semi-annual basis) of a coupon bond is the value of y that solves

$$\text{PRICE} = \frac{C}{y} + \left\{\text{PAR} - \frac{C}{y}\right\}(1+y)^{-2T}. \tag{10.10}$$

Here PRICE is the price of the bond, PAR is the par value, C is the semi-annual coupon payment, and T is the time to maturity in years.

For a zero coupon bond, $C = 0$ and (10.10) becomes

$$\text{PRICE} = \text{PAR}(1+y)^{-2T}. \tag{10.11}$$

Comparing (10.11) and (10.1) we see that for a zero coupon bond, the yield to maturity is the interest rate.

10.4.1 Spot rates

The yield to maturity of a zero coupon bond of maturity n years is called the n-year **spot rate**.

A coupon bond is a bundle of zero coupon bonds, one for each coupon payment and a final one for the principle payment. The component zeros have different maturity dates and therefore different spot rates. The yield to maturity of the coupon bond is a complex "average" of the spot rates of the zeros in this bundle.

Example

Consider the simple example of a one year coupon bond with semi-annual coupon payments of \$40 and a par of \$1000. Suppose that the one-half year spot rate is 5%/year and the one year spot rate is 6%/year. Think of the coupon bond as being composed of two zero coupon bonds, one with $T = 1/2$ and a par of 40 and the second with $T = 1$ and a par of 1040. The price of the bond is the sum of the prices of these two zeros. Applying (10.11) twice to obtain the prices of these zeros and summing, we obtain the price of the zero coupon bond.

$$\frac{40}{1.025} + \frac{1040}{(1.03)^2} = 1019.32.$$

The yield to maturity on the coupon bond is the value of y that solves

$$\frac{40}{1+y} + \frac{1040}{(1+y)^2} = 1019.32.$$

The solution is $y = .0299$. This is the rate per half-year. Thus, the annual yield to maturity is twice .0299 or 5.98%/year. Notice that the yield to maturity is very close to the one-year spot rate rather than close to the arithmetic average of the half-year and one year spot rates. Why do you think this is true?

General formula

Here is a formula that that generalizes the example above. Suppose that a coupon bond pays semi-annual coupon payments of C, has a par value of PAR, and has T years until maturity. Let $r_1, r_2, \ldots, r_{2T}$ be the half-year spot

rates for zero-coupon bonds of maturities $1/2, 1, 3/2, \ldots, T$ years. Then the yield to maturity of the coupon bond is the value of y on half-year basis solves:

$$\begin{aligned}\frac{C}{1+r_1} + \frac{C}{(1+r_2)^2} + \cdots + \frac{C}{(1+r_{2T-1})^{2T-1}} + \frac{\text{PAR}+C}{(1+r_n)^{2T}} \\ = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \cdots + \frac{C}{(1+y)^{2T-1}} + \frac{\text{PAR}+C}{(1+y)^{2T}}.\end{aligned} \tag{10.12}$$

The left hand side of equation (10.12) is the price of the coupon bond, and the yield to maturity is the value of y that makes the right hand side of (10.12) equal to the price.

10.5 Term structure

On January 26, 2001, the *Ithaca Journal* stated that 1-year T-bill rate was 4.83% and the 30-year Treasury bond rate was 6.11%. This is typical—short and long term rates usually do differ. Such differences can be seen in Figure 10.5 of Campbell et al. or Figure 15.7 of Bodie, Kane, and Marcus (1999).

Often short term rates are lower than long-term rates. This makes sense since long term bonds are riskier. Long term bond prices fluctuate more with interest rate changes and these bonds are often sold before maturity. In contrast, a 90-day or even 1-year T-bill is often keep to maturity and so is really a risk-free "fixed income security." However, during periods of very high short-term rates, the short-term rates may be higher than the long term rates. The reason is that the market believes that rates will return to historic levels and no one will commit to the high interest rate for, say, 20 or 30 years. Figure 10.2 shows weekly values of the 30-day, 10-year, and 30-year Treasury rates from 1970 to 1993, inclusive. Notice that the 30-day rate is more volatile that the longer term rates and is usually less than them. However, in the early 1980 when interest rates were very high, the short-term rates were higher than the long-term rates. These data were taken from the Federal Reserve Bank of Chicago's web site.

On April 12, 2002, the 90-day T-bill rate was 1.64%, much lower than a year ago, but the 30-year rate was 5.67%, only somewhat lower than a year ago.

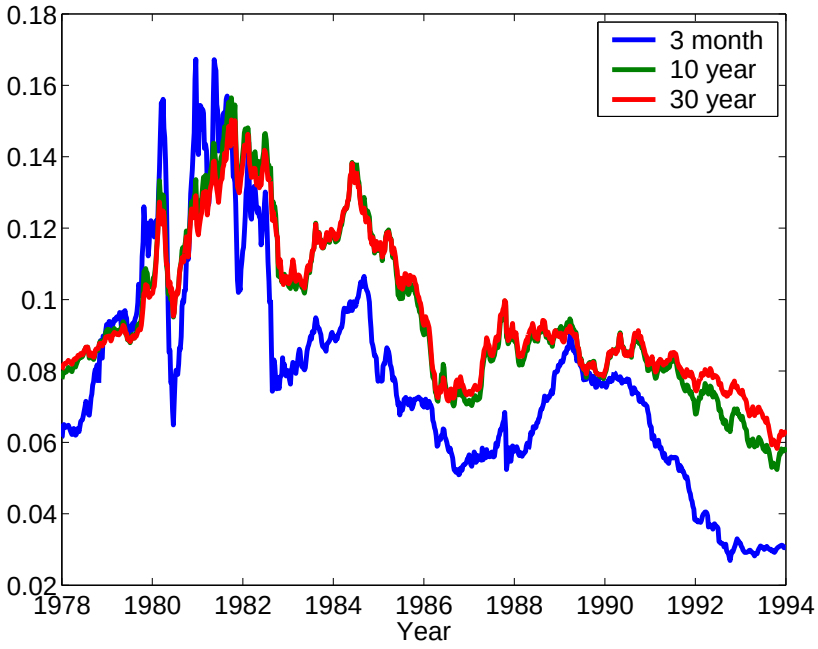

Figure 10.2: *Treasury rates of three maturities. Weekly times series. The data were taken from the web site of the Federal Reserve Bank of Chicago.*

The **term structure** of interest rates is a description of how, *at a given time*, yield to maturity depends on maturity. Term structure for all maturities up to n years can be described by any one of the following:

- prices of zero coupon bonds of maturities 1-year, 2-years, ..., n-years denoted here by $P(1), P(2), \ldots, P(n)$
- spot rates (yields of maturity of zero coupon bonds) of maturities 1-year, 2-years, ... , n-years denoted by $y_1, \ldots, y_n$
- forwards rates $r_1, \ldots, r_n$

As will be seen below, each of the sets

- $\{P(1), \ldots, P(n)\}$
- $\{y_1, \ldots, y_n\}$,

and

- $\{r_1, \ldots, r_n\}$

can be computed from either one of the other sets. For example, (10.14) gives $\{P(1), \ldots, P(n)\}$ in terms of $\{r_1, \ldots, r_n\}$. Also, equations (10.15) and (10.16) give $\{y_1, \ldots, y_n\}$ in terms of $\{P(1), \ldots, P(n)\}$ or $\{r_1, \ldots, r_n\}$, respectively.

Term structure can be described by breaking down the time interval between the present time and the maturity time of a bond into short time segments with a constant interest rate within each segment, but with interest rates varying between segments. For example, a 3-year loan can be considered as three consecutive 1-year loans.

Example:

As an illustration, suppose that the three 1-year loans have the forward interest rates listed in Table 10.1.

Using the forward rates in Table 10.1, we see that a par \$1000 1-year zero would sell for

$$\frac{1000}{1+r_1} = \frac{1000}{1.06} = \$943.40 = P(1).$$

Year (i)	Interest rate (r_i)
1	6%
2	7%
3	8%

Table 10.1: *Forward interest rate example*

A par \$1000 2-year zero would sell for

$$\frac{1000}{(1+r_1)(1+r_2)} = \frac{1000}{(1.06)(1.07)} = \$881.68 = P(2).$$

A par \$1000 3-year zero would sell for

$$\frac{1000}{(1+r_1)(1+r_2)(1+r_3)} = \frac{1000}{(1.06)(1.07)(1.08)} = 816.37 = P(3).$$

The general formula for the present value of \$1 paid n periods from now is

$$\frac{1}{(1+r_1)(1+r_2)\cdots(1+r_n)}. \tag{10.13}$$

Here r_i is the **forward interest rate** during the ith period. By "forward rate" we mean the price for that period that is agreed upon *now*.

Letting $P(n)$ be the price of an n-year par \$1000 zero-coupon bond, $P(n)$ is \$1000 times the discount factor in (10.13), that is,

$$P(n) = \frac{1000}{(1+r_1)\cdots(1+r_n)}. \tag{10.14}$$

Back to the example:

Let's look at the yields to maturity. For a 1-year zero, the yield to maturity y_1 solves

$$\frac{1000}{(1+y_1)} = 993.40,$$

which implies that $y_1 = .06$. Nothing surprising here, since $r_1 = .06$! For a 2-year zero, the yield to maturity is y_2 that solves

$$\frac{1000}{(1+y_2)^2} = 881.68.$$

Thus,

$$y_2 = \sqrt{\frac{1000}{881.68}} - 1 = .0650.$$

It is easy to show that y_2 is also given by

$$y_2 = \sqrt{(1+r_1)(1+r_2)} - 1 = \sqrt{(1.06)(1.07)} - 1 = .0650$$

For a 3-year zero, the yield to maturity y_3 solves

$$\frac{1000}{(1+y_3)^3} = \frac{1000}{881.68}.$$

Also,

$$y_3 = \{(1+r_1)(1+r_2)(1+r_3)\}^{1/3} - 1 = \{(1.06)(1.07)(1.08)\}^{1/3} - 1 = .0700,$$

or, more precisely .069969. Thus, $(1+y_3)$ is the geometric average of 1.06, 1.07, and 1.08 and approximately equal to their arithmetic average.

Recall that $P(n)$ is the price of a par \$1000 n-year zero coupon bond. The general formulas for the yield to maturity y_n of an n-year zero are

$$y_n = \left\{\frac{1000}{P(n)}\right\}^{1/n} - 1, \tag{10.15}$$

and

$$y_n = \{(1+r_1)\cdots(1+r_n)\}^{1/n} - 1. \tag{10.16}$$

Equations (10.15) and (10.16) give the yields to maturity in terms of the bond prices and forward rates, respectively. Also,

$$P(n) = \frac{1000}{(1+y_n)^n}, \tag{10.17}$$

which gives $P(n)$ in terms of the yield to maturity.

As mentioned before, interest rates for future years are called **forward rates**. A forward contract is an agreement to buy or sell an asset at some fixed future date at a fixed price. Since $r_2, r_3, \ldots$ are rates at future dates that are fixed now when a long-term bond is purchased, they are forward rates.

The general formula for determining forward rates from yields to maturity is

$$r_1 = y_1, \tag{10.18}$$

maturity	price
1 year	\$920
2 year	\$830
3 year	\$760

Table 10.2: *Bond price example*

and

$$r_n = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}} - 1. \tag{10.19}$$

Now suppose that we only observed bond prices. Can we calculate yields to maturity and forward rates? The answer is "yes, using (10.15) and then (10.19)."

Example:

Suppose that 1, 2, and 3-year par 1000 zeros are priced as Table 10.2. Then using (10.15), the yields to maturity are:

$$y_1 = \frac{1000}{920} - 1 = .087,$$

$$y_2 = \left\{\frac{1000}{830}\right\}^{1/2} - 1 = .0976,$$

$$y_3 = \left\{\frac{1000}{760}\right\}^{1/3} - 1 = .096,$$

Then, using (10.18) and (10.19)

$$r_1 = y_1 = .087,$$

$$r_2 = \frac{(1+y_2)^2}{(1+y_1)} - 1 = \frac{(1.0976)^2}{1.0876} - 1 = .108,$$

and

$$r_3 = \frac{(1+y_3)^3}{(1+y_2)^2} - 1 = \frac{(1.096)^3}{(1.0976)^2} - 1 = .093.$$

The formula for finding r_n from the prices of zero coupon bonds is

$$r_n = \frac{P(n-1)}{P(n)} - 1, \tag{10.20}$$

which can derived from

$$P(n) = \frac{1000}{(1+r_1)(1+r_2)\cdots(1+r_n)},$$

and

$$P(n-1) = \frac{1000}{(1+r_1)(1+r_2)\cdots(1+r_{n-1})}.$$

To calculate r_1 using (10.20), we need $P(0)$, the price of a 0-year bond, but $P(0)$ is simply the par value. (Trivially, a bond which must be be paid back immediately is worth exactly its par value.)

Example:

Thus, using (10.20)

$$r_1 = \frac{1000}{920} - 1 = .0870,$$

$$r_2 = \frac{920}{830} - 1 = .1084,$$

and

$$r_3 = \frac{830}{760} - 1 = .0921.$$

10.6 Continuous compounding

Now we will assume continuous compounding with forward rates $r_1, \ldots, r_n$. We will see that the use of continuous compounding rates simplifies the relationships between the forward rates, the yields to maturity, and the prices of zero coupon bonds.

If $P(n)$ is the price of a \$1000 par n-year zero coupon bond, then

$$P(n) = \frac{1000}{\exp(r_1 + r_2 + \cdots + r_n)}. \tag{10.21}$$

Therefore,

$$\frac{P(n-1)}{P(n)} = \frac{\exp(r_1 + \cdots + r_n)}{\exp(r_1 + \cdots + r_{n-1})} = \exp(r_n), \tag{10.22}$$

and

$$\log\left\{\frac{P(n-1)}{P(n)}\right\} = r_n.$$

The yield to maturity of an n-year zero coupon bond solves the equation

$$P(n) = \frac{1000}{\exp(ny_n)},$$

and is easily seen to be

$$y_n = (r_1 + \cdots + r_n)/n.$$

Therefore, $\{r_1, \ldots, r_n\}$ is easily found from $\{y_1, \ldots, y_n\}$ by the relationship

$$r_1 = y_n,$$

and

$$r_n = ny_n - (n-1)y_{n-1} \quad \text{for} \quad n > 1.$$

Example:

Using the prices in Table 10.2 (converted from par 1000 to par 1) we have $P(1) = .930$, $P(2) = .850$, and $P(3) = .760$. Therefore,

$$r_1 = \log\left\{\frac{1}{.930}\right\} = .0725,$$

$$r_2 = \log\left\{\frac{.930}{.850}\right\} = .0899,$$

and

$$r_3 = \log\left\{\frac{.850}{.760}\right\} = .1119.$$

Also,

$$y_1 = r_1 = .0725,$$

$$y_2 = (r_1 + r_2)/2 = .0813,$$

and

$$y_3 = (r_1 + r_2 + r_3)/3 = .0915.$$

10.7 Continuous forward rates

So far, we have assumed that forward interest rates vary from year to year, but that these rates are constant within each year. This assumption is, of course, unrealistic. The forward rates should be modeled as a function

varying continuously in time, rather than as a function that is constant for one year at a time. Also, it is unrealistic to assume a fixed starting time for all bonds with all maturities some integer number of years from this date. In fact, bonds are issued throughout the year and bonds of many maturities are on the market.

To specify the term structure in a realistic way, we will assume that there is a function $r(t)$ called the **forward rate function** such that the current price of a zero coupon bond of maturity T and with par equal to 1 is given by

$$P(T) = \exp\left\{-\int_0^T r(t)dt\right\}. \tag{10.23}$$

Formula (10.23) is a generalization of formula (10.21). To appreciate this, suppose that

$$r(t) = r_k \quad \text{for} \quad k-1 < t \leq k.$$

With this piecewise constant r, for any integer T, we have

$$\int_0^T r(t)dr = r_1 + r_2 + \ldots + r_T,$$

so that

$$\exp\left\{-\int_0^T r(t)dt\right\} = \exp\left\{-(r_1 + \cdots + r_T)\right\}$$

and therefore (10.21) agrees with (10.23) in this special situation. However, (10.23) is a more general formula since it applies to non-integer T and to arbitrary $r(t)$, not only piecewise constant functions.

The yield to maturity of a bond with maturity date T is defined to be

$$y_T = \frac{1}{T}\int_0^T r(t)\,dt. \tag{10.24}$$

Think of (10.24) as the average of $r(t)$ over the interval $0 \leq t \leq T$. From (10.23) it follows that

$$P(T) = \exp\{-Ty_T\},$$

so that bond price is the same as it would be if there were a constant forward interest rate equal to y_T.

Example: Suppose the forward rate is $r(t) = .03 + .0005t$. Find $r(15)$, y_{15}, and $P(15)$.

Answer:

$$r(15) = .03 + (.0005)(15) = .0375.$$

$$y_{15} = (15)^{-1} \int_0^{15} (.03 + .0005t)dt = (15)^{-1}(.03t + .0005t^2/2)\Big|_0^{15} = .03375.$$

$$P(15) = \exp(-15y_{15}) = \exp\{-(15)(.03375)\} = \exp(.5055) = .6028.$$

The price, $P(T)$ and forward rate function, $r(t)$, in formula (10.23) depend on the current time, which is taken to be zero in that formula. However, we could be interested in how the price and forward rate function change over time. In that case we let $P(s,T)$ be the price at time s of a zero coupon bond, with a par of \$1, maturing at time T. Then

$$P(s,T) = \exp\left\{-\int_s^T r(s,t)dt\right\}. \tag{10.25}$$

Since $r(t)$ and $P(t)$ in (10.23) are $r(0,t)$ and $P(0,t)$ in our new notation, (10.23) is the special case of (10.25) with $s = 0$.

10.8 Estimation of a continuous forward rate

We will now assume that $s = 0$ and return to the simpler notation of $r(t)$ and $P(t)$. In practice, the forward rate function $r(t)$ is unknown. Only bond prices are known. However, we can estimate $r(t)$ from the bond prices using nonlinear regression. An example of estimating $r(t)$ was given in Section 6.5 assuming that $r(t)$ was constant and using as data the prices of zero coupon bonds of different maturities. In this section, we will estimate $r(t)$ without assuming it is constant.

Suppose that $r(t)$ is linear, for example, that

$$r(t) = \beta_0 + \beta_1 t.$$

Then

$$\int_0^T r(t)dt = \beta_0 T + \beta_1 T^2/2.$$

The nonlinear regression model states that the price of a bond with a par of \$1 and maturity T_i is

$$P(T_i) = \exp\left\{-\left(\beta_0 T_i + \frac{\beta_1 T_i^2}{2}\right)\right\} + \epsilon_i, \tag{10.26}$$

where ϵ_i is an "error" due to problems such as prices being somewhat stale[3] and bid-ask spread.[4]

We will now look at an example using data on US STRIPS, a type of zero coupon bonds. STRIPS is an acroynm for "separate trading of registered interest and principal of securities." The interest and principal on Treasury bills, notes, and bonds are traded separated through the Federal Reserve's book-entry system, in effect creating zero coupon bonds by repackage coupon bonds. [5]

The data is from December 31, 1995. The prices are given as a percentage of par. For example, a price of 95 means that the bond is selling at 95% of its par value, e.g., for \$950 if the par value is \$1000.

Here are the first five records of data. The variable are current date, maturity date, price, and record number (order of the observation in this data set). You can see that the current data is always Dec, 31, 1995. The first bond matures on Feb 15, 1996 and is selling at 99.393% of par.

```
19951231  19960215  99.393  1
19951231  19960815  96.924  2
19951231  19970215  94.511  3
19951231  19970815  92.070  4
19951231  19980215  89.644  5
```

Price is plotted against maturity in years in Figure 10.3. Maturities are nearly equally spaced from 0 to 30 years. We can see that the price drops smoothly with maturity and that there is not much noise in the data.

[3] Prices are "stale" because they can be determined only at the time of the last sale.

[4] A bond dealer will buy bonds at the bid price and sell them at the ask price which is slightly higher than the bid price. The difference is called the bid-ask spread and covers the trader's administrative costs and profit.

[5] Jarrow (1996, p. 11).

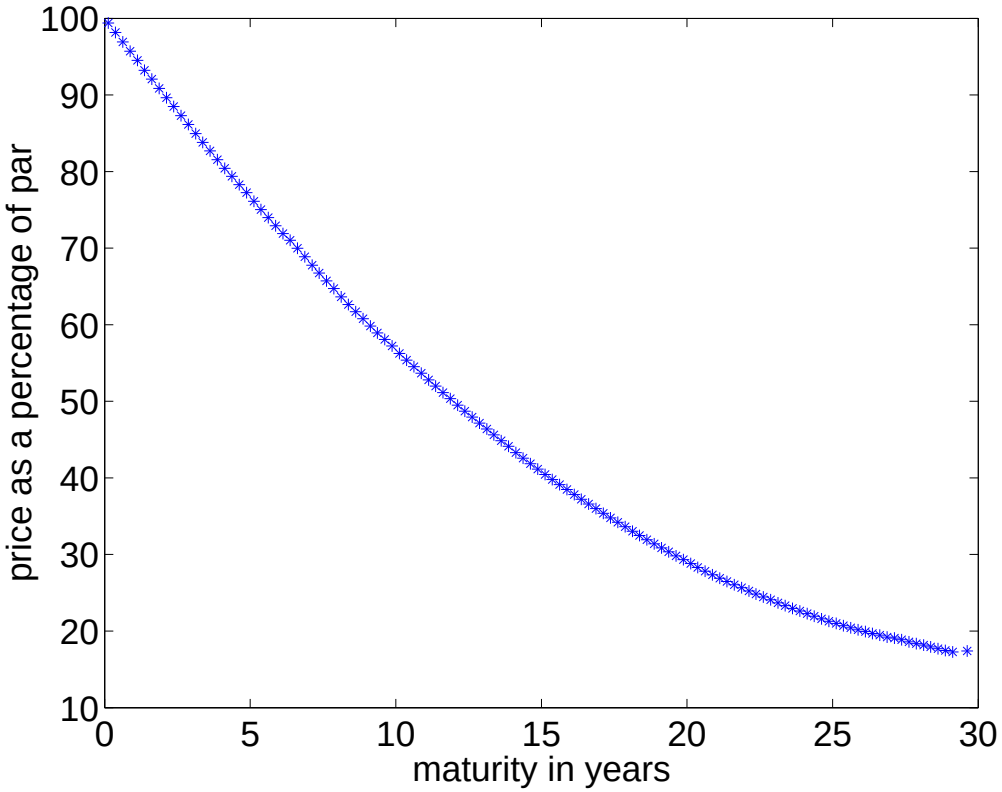

Figure 10.3: *Prices of the U.S. STRIPS.*

Here is a SAS program to fit this model to the US STRIPS data. In the search for good starting values for the iterative search, the parameter β_0 varies on a grid from .02 to .09. The idea here is that β_0 is the "constant" in the interest rate model. The interest rates should be close to β_0 and .02 to .09 is the typical range of interest rates. The parameter β_1 determines how much the interest rates will deviate from a constant value. We expect this deviation to be small and that it could be either positive or negative. Therefore, the initial grid of values of β_1 is $-.01$ to .01 by .005.

```
options linesize=64;
data USstrips ;
infile 'c:\courses\or473\data\strips.dat' ;
input today maturity price time2mat;
time2mat = time2mat/365 ;
title 'Nonlinear regression using US STRIPS bond data';
proc nlin ;
parm beta0=.02 to .09 by .005 beta1= -.01 to .01 by .005 ;
model price = 100*exp(-beta0*time2mat - (beta1*(time2mat)**2)/2 ) ;
run ;
```

The first bit of output is the sum of squares computed on the grid of β_0 and β_1 values. Most of this output is omitted here.

```
          NL regression using US STRIPS data, linear          1
                              09:24 Thursday, April 4, 2002

                  The NLIN Procedure
                     Grid Search
                Dependent Variable price

                                          Sum of
               beta0       beta1        Squares

              0.0200     -0.0100      2.9019E8
              0.0250     -0.0100      2.1996E8
              0.0300     -0.0100      1.6673E8
              0.0350     -0.0100      1.2637E8
              0.0400     -0.0100      95772757
              0.0450     -0.0100      72567610
```

The next bit of output is from the iterative phase of the nonlinear least-squares estimation. The computations converge to the least-squares estimates $\widehat{\beta}_0 = .0533$ and $\widehat{\beta}_1 = .000363$.

```
NL regression using US STRIPS data, linear          3
                      09:24 Thursday, April 4, 2002

          The NLIN Procedure
           Iterative Phase
        Dependent Variable price
          Method: Gauss-Newton

                                    Sum of
Iter       beta0       beta1      Squares

   0      0.0600           0        503.9
   1      0.0534    0.000705        109.2
   2      0.0533    0.000725        108.6
   3      0.0533    0.000725        108.6
   4      0.0533    0.000725        108.6

NOTE: Convergence criterion met.
```

The final bit of output is the results of the nonlinear least-squares estimation.

```
          Estimation Summary

  Method                 Gauss-Newton
  Iterations                        4
  R                          1.212E-6
  PPC(beta1)                 6.175E-7
  RPC(beta1)                 0.000023
  Object                     2.187E-9
  Objective                  108.6365
  Observations Read               236
  Observations Used               236
  Observations Missing              0

NOTE: An intercept was not specified for this model.
```

Source	DF	Sum of Squares	Mean Square	F Value	Approx Pr > F
Regression	2	670317	335159	721922	<.0001
Residual	234	108.6	0.4643		
Uncorrected Total	236	670426			
Corrected Total	235	141321			

NL regression using US STRIPS data, linear 4
09:24 Thursday, April 4, 2002

The NLIN Procedure

Parameter	Estimate	Approx Std Error	Approximate 95% Confidence Limits	
beta0	0.0533	0.000230	0.0529	0.0538
beta1	0.000725	0.000026	0.000675	0.000775

Approximate Correlation Matrix

	beta0	beta1
beta0	1.0000000	-0.9269206
beta1	-0.9269206	1.0000000

Note that the confidence interval for β_1 does not contain 0. Therefore, we can conclude with 95% confidence that β_1 is not 0 and the forward function is not constant. If the forward curve is linear, then it appears to be increasing since $\beta_1 > 0$ (with 95% confidence). However, there is no compelling reason to assume that the forward function is linear. There may be curvature to this function. To check for curvature, we can fit a quadratic model for the forward rate function. That is, we fit the model

$$r(t) = \beta_0 + \beta_1 t + \beta_2 t^2,$$

which gives us the new regression model

$$P(T_i) = \exp\left\{-\left(\beta_0 T_i + \frac{\beta_1 T_i^2}{2} + \frac{\beta_2 T_i^3}{3}\right)\right\} + \epsilon_i,$$

Here is the SAS program to fit a quadratic forward rate function.

```
options linesize=64;
data USStrips ;
infile 'c:\courses\or473\data\strips.dat' ;
input today maturity price time2mat;
time2mat = time2mat/365 ;
run ;
title 'NL regression using US Strips bond, quadratic';
proc nlin ;
parm beta0 = .02 to .09 by .005 beta1 = -.01 to .01 by .005
     beta2 = -.01 to .01 by .005 ;
model price = 100*exp(-beta0*time2mat - (beta1*(time2mat)**2)/2
     - (beta2*(time2mat)**3)/3) ;
run ;
```

Here are selected parts of the output with the results of the iterative estimation phase first.

```
         NL regression using US Strips bond, quadratic         6
                                11:01 Thursday, April 4, 2002

                      The NLIN Procedure
                       Iterative Phase
                    Dependent Variable price
                      Method: Gauss-Newton

                                                      Sum of
      Iter       beta0       beta1       beta2      Squares

         0      0.0600           0           0       503.9
         1      0.0471     0.00249    -0.00008     11.3192
         2      0.0474     0.00245    -0.00008     11.1848
         3      0.0474     0.00245    -0.00008     11.1848


NOTE: Convergence criterion met.


                      Estimation Summary

              Method                 Gauss-Newton
              Iterations                        3
              R                          5.51E-6
```

PPC(beta2)	1.48E-6
RPC(beta2)	0.00011
Object	2.683E-7
Objective	11.18479
Observations Read	236
Observations Used	236
Observations Missing	0

NOTE: An intercept was not specified for this model.

Source	DF	Sum of Squares	Mean Square	F Value	Approx Pr > F
Regression	3	670415	223472	4655328	<.0001
Residual	233	11.1848	0.0480		
Uncorrected Total	236	670426			
Corrected Total	235	141321			

NL regression using US Strips bond, quadratic 7
11:01 Thursday, April 4, 2002

The NLIN Procedure

Parameter	Estimate	Approx Std Error	Approximate 95% Confidence Limits	
beta0	0.0474	0.000151	0.0471	0.0477
beta1	0.00245	0.000039	0.00237	0.00252
beta2	-0.00008	1.706E-6	-0.00008	-0.00007

Approximate Correlation Matrix

	beta0	beta1	beta2
beta0	1.0000000	-0.9486321	0.8748783
beta1	-0.9486321	1.0000000	-0.9783902
beta2	0.8748783	-0.9783902	1.0000000

The confidence interval for β_2 contains only negative numbers indicating that the with 95% confidence we can conclude that β_2 is negative. This is strong evidence that the forward rate curve is not linear, but rather is

concave.[6]

Figure 10.4 shows the fitted constant, linear, and quadratic forward curves. There is also a fitted curve using a spline that will be discussed later. This figure was plotted using a MATLAB program. That program also computed AIC and found AIC to be 174.2, −177.1, and −707.9 for constant, linear, and quadratic rate curves, respectively. Since small values of AIC indicate a good model, AIC chooses the quadratic model over the constant and linear models.

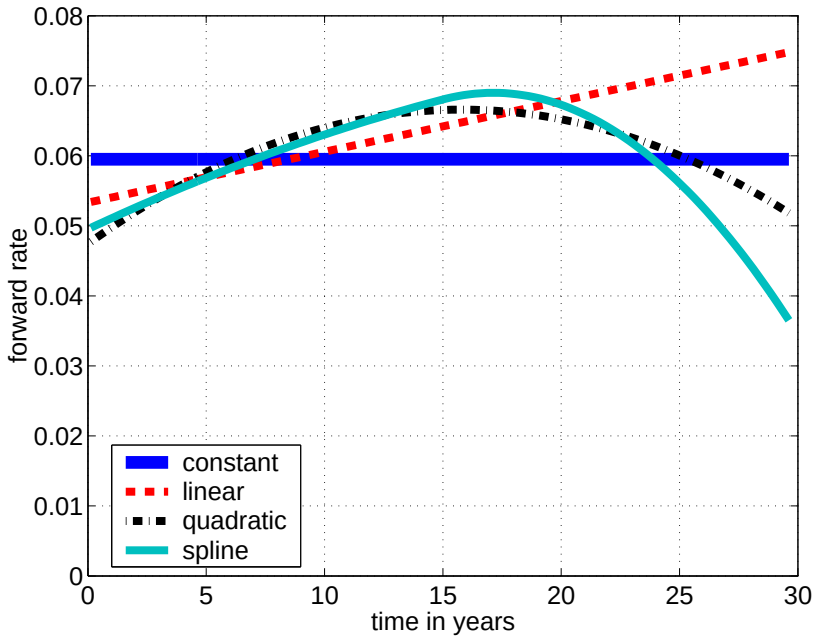

Figure 10.4: *Polynomial estimates of forward rates of US Treasury bonds.*

Of course, there is no guarantee that the forwarded curve is really quadratic, only that a quadratic model fits better than a constant or linear function. Perhaps a higher degree polynomial or a non-polynomial function would

[6] A function is concave if its second derivative is always non-positive. This means that its first derivative is always constant or decreasing. For example, a parabola that opens downward is concave.

be an even better model. A flexible way of modeling a arbitrary smooth curve is with a spline. A spline is a function that is created by piecing together polynomials. The spline in Figure 10.4 is composed of two quadratic functions, one for $t < 15$ and the other for $t > 15$. The AIC for this model is -1221.1, much smaller than for the quadratic model, indicated that the spline model fits better than any of the polynomial models that were tried before.

The two quadratic functions that make up the spline function are pieced together so that at $t = 15$ the spline function is continuous with a continuous first derivative. This means that the two functions join together without a "gap" that would occur if the spline jumped in value at $t = 15$. Also, they join together without a "kink" that would occur if the first derivative jumped in value at $t = 15$. The second derivative is discontinuous and in fact jumps in value at $t = 15$ causing a change in curvature at $t = 15$. Quadratic functions have a constant second derivative[7] so the second derivative of the spline has a constant value for $t < 15$ and then jumps to a new constant value for $t > 15$. Second derivatives measure curvature, and you can see that the fitted spline in Figure 10.4 has one constant curvature to the left of 15 and a greater constant curvature to the right of 15.

The location, $t = 15$, where the spline changes from one quadratic function to another is called a *knot* of the spline. Putting the knot at $t = 15$ was somewhat arbitrary, though 15 is the midpoint of the range of t.

Ideally we would like the shape of the fitted spline to depend only on the data, not on the number of knots and the locations of the knots. This can be accomplished by using a relatively large number of knots, say 10 to 20, and placing then evenly over the range of t. Figure 10.5 shows spline fits to the STRIPS data with 1, 5, 10, and 20 knots. The one and five knot splines are similar, but the five knot, but the extra flexibility of the five knot spline allows it to fit the data somewhat better. A potential problem with a large number of knots is that the fitted forward curve can oscillate too much. This can be seen in Figure 10.5 when there are 10 or 20 knots, especially the latter case. The problem with 10 or more knots is that the data are "overfit," meaning that there are so many parameters than the random noise in the data is being fit by the spline.

[7] Recall that $a + bx + cx^2$ has second derivative equal to $2c$.

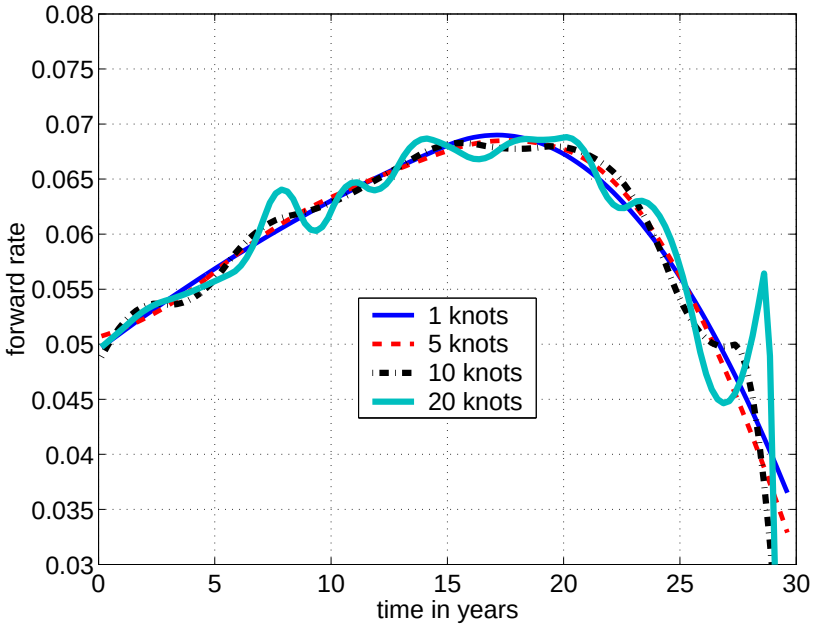

Figure 10.5: *Splines with 1, 5, 10, or 20 knots fit to US STRIPS data.*

The overfitting problem is solved by Jarrow, Ruppert, and Yu (2001) by penalizing large jumps in the second derivative. This penalization of overfitting prevents the fitted forward curve from being too wiggly. The overfitting penalty is analogous to the penalties discused in Sections 4.7.1 and 6.4 on the number of parameter in on model that are incorporated into the AIC and C_p criteria. Figure 10.6 shows estimated forward rate curves for US STRIPS. Figure 10.7 shows estimated forward rate curves for AT&T bonds, assuming that the AT&T forward rate is the STRIPS forward rate plus a polynomial "spread." The "spread" is the risk premium on the default risk of corporate bonds. The default risk on the STRIPS is assumed to be zero since these are US Treasury bonds.

There are only five AT&T bonds so it is impossible to estimate the forward curve for AT&T bonds using only AT&T bond prices. By assumimg a simple model for the spread, it is possible to use the US STRIPS data to help determine the AT&T forward rate. These two figures come from Jarrow, Ruppert, and Yu (2001) where the overfitting penalty and the use of

US STRIPS prices to estimate corporate bond term structure is explained in more detail.

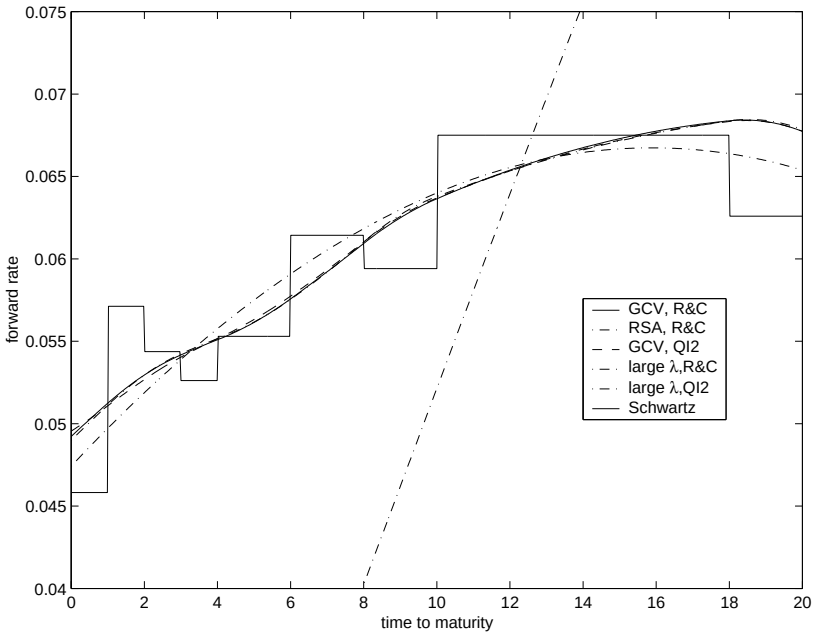

Figure 10.6: *Spline estimates of forward rates of US Treasury bonds.*

10.9 Summary

Introduction

- buy a bond = making a loan to the company
  - corporation is obligated to pay back the principle and interest (unless it defaults)
  - you receive a fixed stream of income
  - bonds are called "fixed-income" securities

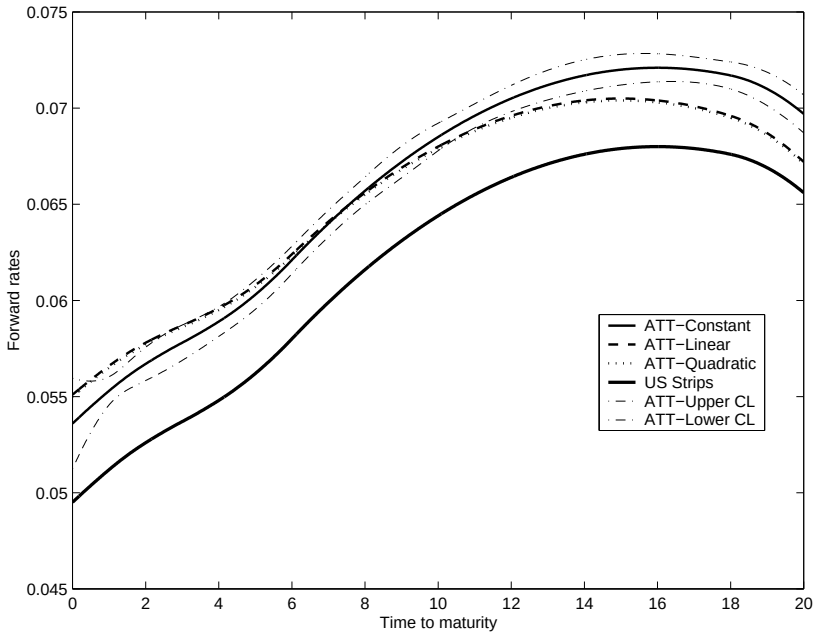

Figure 10.7: *Spline estimates of forward rates of AT&T bonds*

– for long term bond your income is guaranteed only if you keep the bond to maturity

Zero coupon bonds

- Zero-coupon bonds pay no principle or interest until maturity
- zero-coupon bond = pure discount bond
- par value is the payment made to the bond holder at maturity
- a zero sells for less than par
- **Example:** 20-year zero
  - par value of $1000
  - interest 6% compounded every six months ⇒ price is

$$\frac{\$1000}{(1.03)^{40}} = \$306.56,$$

Risk due to interest rate changes

- bond prices fluctuate with the interest rate
  - **Example:** assume semi-annual compounding
  - you just bought the zero for \$306.56
    - six months later the interest rate increased to 7%
  - price would now be
$$\frac{\$1000}{(1.035)^{39}} = \$261.41$$
  - investment would drop by (\$306.56 − \$261.41) = \$45.15
  - return of
$$\frac{-45.15}{306.56} = -14.73\%$$
    for a half-year or −29.46% per year
  - however, if the interest rate remains unchanged then the bond is worth
$$\frac{\$1000}{(1.03)^{39}} = (1.03)(\$306.56)$$
    - 3%/half-year return

Coupon bonds

- coupon bonds make regular interest payments
- consider a 20-year coupon bond with a par value of \$1000 and 6% annual interest with semi-annual coupon payments
  - coupon payment will be \$30
  - bond holder receives 40 payments of \$30
    - plus a principle payment of \$1000 after 20 years
  - present value of all payments, with discounting at the 6% annual rate (3% semi-annual), equals \$1000:
$$\sum_{t=1}^{40} \frac{30}{(1.03)^t} + \frac{1000}{(1.03)^{40}} = 1000.$$

- **General formula:**
  - Notation
    - PAR = par value
    - C = coupon payment
    - T = maturity
    - r = interest rate per half-year
  - bond price =

$$\sum_{t=1}^{2T} \frac{C}{(1+r)^t} + \frac{\text{PAR}}{(1+r)^{2T}}$$
$$= \frac{C}{r} + \left\{\text{PAR} - \frac{C}{r}\right\}(1+r)^{-2T}$$

- **Yield to maturity**
- **Example:** a bond with $T = 30$ and $C = 40$ is selling for \$1200,
- bond selling at par $\Rightarrow$ interest rate = .04/half-year (= .08/year).
  - 4%/half-year rate = **coupon rate**.
- but *not* selling at par $\Rightarrow$ if you purchase the bond at \$1200 you will make *less* than 8% per year
- two problems
  - coupon payments are \$40 or 40/1200 = 3.333%/half-year of the \$1200 investment
    - 3.333% is called the **current yield**
  - at maturity you only get back \$1000 of the \$1200 investment
  - **yield to maturity** = the average rate of return
- **Spot rates**
  - The yield to maturity of a zero coupon bond of maturity n years is called the n year **spot rate**.
  - A coupon bond is a bundle of zeros, each with a different maturity and therefore a different spot rate
    - the yield to maturity of a coupon bond is a complex "average" of these different spot rates

Term structure of interest rates

- term structure is description of how, *at a given time*, yield to maturity depends on maturity
- term structure for all maturities up to n years can be described by any one of the following sets:
  - prices of zero coupon bonds of maturities 1-year, 2-years, ..., n-years denoted here by $P(1), P(2), \ldots, P(n)$
  - spot rates (yields of maturity of zero coupon bonds) of maturities 1-year, 2-years, ... , n-years denoted by $y_1, \ldots, y_n$
  - forwards rates $r_1, \ldots, r_n$
- each of the above sets can be computed from either of the other sets.

Continuous compounding

- continous compounding simplifies the relationships between
  - forward rates
  - yields to maturity of zeros (spot rates)
  - prices of zeros
- prices from forward rates:

$$P(1) = \frac{1000}{\exp(r_1)},$$

$$P(2) = \frac{1000}{\exp(r_1)\exp(r_2)},$$

  etc., so that

$$P(n) = \frac{1000}{\exp(r_1 + r_2 + \cdots + r_n)}.$$

- forward rates from prices:

$$\frac{P(n-1)}{P(n)} = \frac{\exp(r_1 + \cdots + r_n)}{\exp(r_1 + \cdots + r_{n-1})} = \exp(r_n)$$

  $\Rightarrow$

$$r_n = \log\left\{\frac{P(n-1)}{P(n)}\right\}$$

- yield to maturity y_n solves

$$P(n) = \frac{1000}{\exp(ny_n)},$$

$\Rightarrow$

$$y_n = (r_1 + \cdots + r_n)/n.$$

- $\{r_1, \ldots, r_n\}$ is easily found from $\{y_1, \ldots, y_n\}$ by:

$$r_1 = y_1,$$

and

$$r_n = ny_n - (n-1)y_{n-1} \quad \text{for} \quad n > 1.$$

10.10 References

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Chapter 11

Resampling and Portfolio Analysis: 4/30/02

11.1 Introduction

Computer simulation is widely used in all areas of operations research and, in particular, applications of simulation to statistics have become very widespread. In this chapter we apply a simulation technique called the "bootstrap" or "resampling" to study the effects of estimation error on portfolio selection. The term "bootstrap" was coined by Bradley Efron and comes from the phrase "pulling oneself up by one's bootstraps" that apparently originated in the 18th century story "Adventures of Baron Munchausen" by Rudolph Erich Raspe (Efron and Tibshirani, 1993). In this chapter, "bootstrap" and "resampling" will be treated as synonymous.

When statistics are computed from a randomly chosen sample, then these statistics are random variables. Students often do not appreciate this fact. After all, what could be random about $\overline{X}$? We just average the data, so what is random? The point is that the sample is only one of many possible samples. Each possible sample gives a possible value of $\overline{X}$. Thus, although we only see one value of $\overline{X}$, it was selected at random from the many possible values. Thus, $\overline{X}$ is a random variable.

Methods of statistical inference such as confidence intervals and hypothesis tests are predicated on the randomness of statistics. For example, the confidence coefficient of a confidence interval tells us the probability

that an interval constructed from a random sample will contain the parameter.

Confidence intervals are usually derived used probability theory. Often, however, the necessary probability calculations are intractable. In that case we can replace theoretical calculations by Monte Carlo simulation.

But how do we simulate sampling from an *unknown* population? The answer, of course, is that we cannot do this exactly. However, a sample is a good representative of the population, and we can simulate sampling from the population by sampling from the sample, which is usually called "resampling."

Each resample has the same sample size, n, as the original sample. The reason for this is that we are trying to simulate the original sampling, so we want the resampling to be as similar as possible to the original sampling. Moreover, the resamples are drawn *with replacement*. Why? The reason is that only sampling with replacement give independent observations, and we want the resamples to be i.i.d. just like the original sample. In fact, if the resamples were drawn without replacement then every resample would be exactly the same as the original sample, so the resamples would show no random variation. This wouldn't be very satisfactory, of course.

The number of resamples taken should, in general, be large. Just how large depends on context and will be discussed more fully later. Sometimes thousands or even tens of thousands of resamples are used. We will let B denote the number of resamples.

There is some good news and some bad news about the bootstrap. The good news is that computer simulation replaces difficult mathematics. The bad news is that resampling is a new concept that takes some time to become comfortable with. The problem is not that resampling is all that conceptually complex. Rather, the problem is that students don't have a great deal of experience with drawing even a single random sample from a population. Resampling is even more complex that that, with two layers of sampling and multiple resamples.

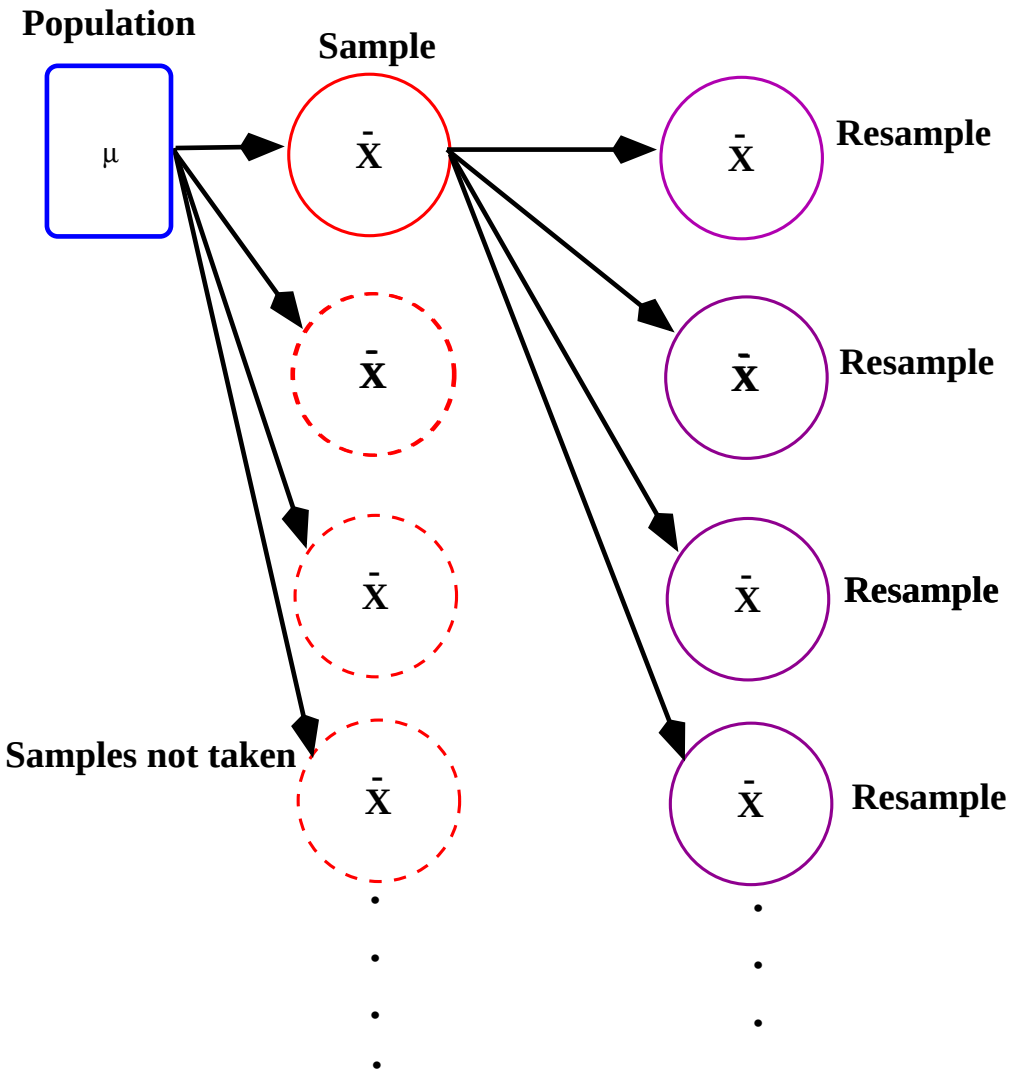

Figure 11.1: *Resampling. There is a single population with mean μ. A single sample is taken and has mean $\overline{X}$. Other samples could have been taken but were not. Thus, $\overline{X}$ is the observed value of a random variable. We want to know the probability distribution of $\overline{X}$ or, perhaps, t. To approximate this distribution, we resample from the sample. Each resample has a mean, $\overline{X}_{boot}$.*

11.2 Confidence intervals for the mean

Before we discuss resampling of the efficient frontier, we will look at a simpler problem. Suppose we wish to construct a confidence interval for the population mean based on a random sample. One starts with the so-called "t-statistic"[1] which is

$$t = \frac{\mu - \overline{X}}{\frac{s}{\sqrt{n}}}. \tag{11.1}$$

The denominator of t, $s/\sqrt{n}$, is just the standard error of the mean so that the denominator estimates the standard deviation of the numerator.

If we are sampling from a normally distributed population, then the probability distribution of t is known to be the t-distribution with $n-1$ degrees of freedom. We will denote by $t_{\alpha/2}$ the $\alpha/2$ upper t-value, that is the $1-\alpha/2$ quantile of this distribution. Thus t in (11.1) has probability $\alpha/2$ of exceeding $t_{\alpha/2}$. Because of the symmetry of the t-distribution, the probability is also $\alpha/2$ than t is less that $-t_{\alpha/2}$.

Therefore, for normally distributed data, the probability is $1-\alpha$ that

$$-t_{\alpha/2} \leq t \leq t_{\alpha/2}. \tag{11.2}$$

Substituting (11.1) into (11.2), after a bit of algebra we find that the probability is $1-\alpha$ that

$$\overline{X} - t_{\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{X} + t_{\alpha/2}\frac{s}{\sqrt{n}}, \tag{11.3}$$

which shows that

$$\overline{X} \pm \frac{s}{\sqrt{n}} t_{\alpha/2}$$

is a $1-\alpha$ confidence interval for μ, assuming normally distributed data.

What if we are not sampling from a normal distribution? In that case, the distribution of t (11.1) will *not* be the t distribution, but rather some other distribution that is not known to us. There are two problems. First, we don't know the distribution of the population. Second, even if the population distribution were known it is a difficult, usually intractable, probability calculation to get the distribution of the t-statistic from the distribution

[1] Actually, t is not quite a statistic since it depends on the unknown μ, whereas a statistic, by definition, is something that depends only on the sample, not on unknown parameters.

of the population. This calculation has only be done for normal populations. Considering the difficulty of these two problems, can we still get a confidence interval? The answer is "yes, by resampling." We can take a large number, say B, resamples from the original sample.

Let $\overline{X}_{\text{boot},b}$ and $s_{\text{boot},b}$ be the sample mean and standard deviation of the bth resample, $b = 1, \ldots, B$. Define

$$t_{\text{boot},b} = \frac{\overline{X} - \overline{X}_{\text{boot},b}}{\frac{s_{\text{boot},b}}{\sqrt{n}}}. \tag{11.4}$$

Notice that $t_{\text{boot},b}$ is defined in the same way as t except for two changes. First, $\overline{X}$ and s in t are replaced by $\overline{X}_{\text{boot},b}$ and $s_{\text{boot},b}$ in $t_{\text{boot},b}$. Second, μ in t is replaced by $\overline{X}$ in $t_{\text{boot},b}$. The last point is a bit subtle, and you should stop to think about it. A resample is taken using the original sample as the population. Thus, for the resample, the population mean is $\overline{X}$!

Because the resamples are independent of each other, the collection $t_{\text{boot},1}$, $t_{\text{boot},2}, \ldots$ can be treated as a random sample from the distribution of the t-statistic. After B values of $t_{\text{boot},b}$ have been calculated, one from each resample, we find the 2.5% and 97.5% percentiles of this collection of $t_{\text{boot},b}$ values. Call these percentiles t_L and t_U. More specifically, we find t_L and t_U as follows. The B values of $t_{\text{boot},b}$ are sorted from smallest to largest. Then we calculate $B\alpha/2$ and round to the nearest integer. Suppose the result is K_L. Then the K_Lth sorted value of $t_{\text{boot},b}$ is t_L. Similarly, let K_U be $B(1 - \alpha/2)$ rounded to the nearest integer and then t_U is the K_Uth sorted value of $t_{\text{boot},b}$ is t_U.

If the original population is skewed, then there is no reason to suspect that the 2.5% percentile is minus the 97.5% percentile, as happens for symmetric populations like the t-distribution. In other words, we do not necessarily expect that $t_L = -t_U$. However, this fact causes us no problem since the bootstrap allows us to estimate t_L and t_U without assuming any relationship between them. Now we replace $-t_{\alpha/2}$ and $t_{\alpha/2}$ in the confidence interval (11.3) by t_L and t_U, respectively. Finally, the bootstrap confidence interval for μ is

$$\left(\overline{X} + t_L \frac{s}{\sqrt{n}},\ \overline{X} + t_U \frac{s}{\sqrt{n}}\right).$$

The bootstrap has solved both problems mentioned above. We do not need to know the population distribution since we can estimate it by the

sample.[2] Moreover, we don't need to calculate the distribution of the t-statistic using probability theory. Instead we can simulate from this distribution.

We will use the notation

$$\text{SE} = \frac{s}{\sqrt{n}}$$

and

$$\text{SE}_{\text{boot}} = \frac{s_{\text{boot}}}{\sqrt{n}}.$$

Example: We start with a very small sample of size six to illustrate how the bootstrap works. The sample is 82, 93, 99, 103, 104, 110, $\overline{X} = 98.50$, and the SE is 4.03.

The first bootstrap sample is 82, 82, 93, 93, 103, 110. In this bootstrap sample, 82 and 93 were sampled twice, 103 and 110 were sampled once, and the other elements of the original sample were not sampled. This happened by chance, of course, since the bootstrap sample was taken at random, with replacement, from the original sample. For this bootstrap sample $\overline{X}_{\text{boot}} = 93.83$, $\text{SE}_{\text{boot}} = 4.57$, and $t_{\text{boot}} = (98.5 - 93.83)/4.57 = 1.02$.

The second bootstrap sample is 82, 103, 110, 110, 110, 110. In this bootstrap sample, 82 and 103 were sampled twice, 110 was sampled four times, and the other elements of the original sample were not sampled. It may seem strange at first that 110 was resample four times. Remember that this event happened by chance and, in fact, is not highly unlikely either and will happen occasionally.[3] For this bootstrap sample $\overline{X}_{\text{boot}} = 104.17$, $\text{SE}_{\text{boot}} = 4.58$, and $t_{\text{boot}} = (98.5 - 104.17)/4.58 = -1.24$.

The third bootstrap sample is 82, 82, 93, 99, 104, 110. For this bootstrap sample $\overline{X}_{\text{boot}} = 95.00$, $\text{SE}_{\text{boot}} = 4.70$, and $t_{\text{boot}} = (98.5 - 95.00)/4.570 = 1.02$.

[2] This is something a bit clever going on here. A sample isn't a probability distribution. What we are doing is creating a probability distribution from the sample by giving each observation in the sample probability $1/n$ where n is the sample size.

[3] The number of times that 110 appears in any resample is a binomial random variable with parameters $p = 1/6$ and $n = 6$. Therefore, the probability that 110 occurs exactly four times in the sample is $\{6!/(4!\,2!)\}(1/6)^4(5/6)^2 = 0.00804$. Of course, there is nothing special about 110 and we might have been surprised if any element of the sample appeared four times in a resample. In fact, however, we should not be surprised. The probability that one of the six elements of the original sample will occur exactly four times in a resample is (6)(.00804) = .0482. Thus, we will see some element of the sample resampled exactly four times in about one bootstrap resample out of 21.

If this example were to continue, more bootstrap samples would be drawn and all bootstrap t values would be saved in order compute quantiles of the bootstrap t values. However, since the sample size is so small, this example is not very realistic and we will not continue it.

Example: Suppose that we have a random sample with a more realistic size of 40 from some population and $\overline{X} = 107$ and $s = 12.6$. Let's find the "normal theory" 95% confidence interval for the population mean μ. With 39 degrees of freedom, $t_{.025} = 2.02$. Therefore, the confidence interval for μ is

$$107 \pm 2.02\frac{12.6}{\sqrt{40}} = (102.97, 111.03).$$

Suppose that we use resampling instead of normal theory and that we use 1,000 resamples. This gives us 1,000 values of $t_{\text{boot},b}$. We rank them from smallest to largest. The 25% percentile is the one with rank 25 = (1000)(.025). Suppose the 25th smallest value of $t_{\text{boot},b}$ is -1.98. The 97.5% percentile is the value of $t_{\text{boot},b}$ with rank 975. Suppose that its value is 2.25. Then $t_L = -1.98$, $t_U = 2.25$, and the 95% confidence interval for μ is

$$\left(107 - 1.98\frac{12.6}{\sqrt{40}},\ 107 + 2.25\frac{12.6}{\sqrt{40}}\right) = (103.06,\ 111.48).$$

Example: Log-returns for MSCI-Argentina MSCI-Argentina is the Morgan Stanley Capital Index for Argentina and is roughly comparable to the S&P 500 for the US. The log-returns for this index from January 1988 to January 2002, inclusive, are used in this example. A normal plot of these results is found in Figure 11.4 in the next section, where the data are discussed more thoroughly. There is evidence of non-normality, in particular, that the log-returns are heavy-tailed, especially the left tail.

The bootstrap was implemented with $B = 10,000$. The t-values were $t_L = -1.93$ and $t_U = 1.98$. To assess the Monte Carlo variability, the bootstrap was repeated two more times with the results first that $t_L = -1.98$ and then that $t_U = 1.96$ and the $t_L = -1.94$ and $t_U = 1.94$. We see that $B = 10,000$ gives reasonable accuracy but that the third significant digit is still uncertain. Also, using normal theory, $t_L = -t_{.025} = -1.974$ and $t_U = t_{.025} = 1.974$, which are similar to the bootstrap values that do not assume normality. Therefore, the use of the bootstrap in this example confirms that the normal theory confidence interval is satisfactory. In other

example, particularly with strongly skewed data and small sample sizes, the normal theory confidence interval will be less satisfactory.

11.3 Resampling the efficient frontier

One application of optimal portfolio selection is to allocation of capital to different market segments. For example, Michaud (1998) discusses a global asset allocation problem where capital must be allocated to "U.S. stocks and government/corporate bonds, Euros, and the Canadian, French, German, Japanese, and U.K. equity markets." Here we look at a similar example where we allocate capital to the equity markets of ten different countries. Monthly log-returns for these markets were calculated from:

- 1 = MSCI Hong Kong
- 2 = MSCI Singapore
- 3 = MSCI Brazil
- 4 = MSCI Argentina
- 5 = MSCI UK
- 6 = MSCI Germany
- 7 = MSCI Canada
- 8 = MSCI France
- 9 = MSCI Japan
- 10 = S&P 500

The data are in the file "countries.txt" on the course web site abd came from Datastream. "MSCI" means "Morgan-Stanley Capital Index." The data are from January 1988 to January 2002, inclusive, so there are 169 months (14 years and one month) of data.

Figure 11.2 shows time series plots of the returns for each of these countries. The plots look stationary though there may be GARCH effects.

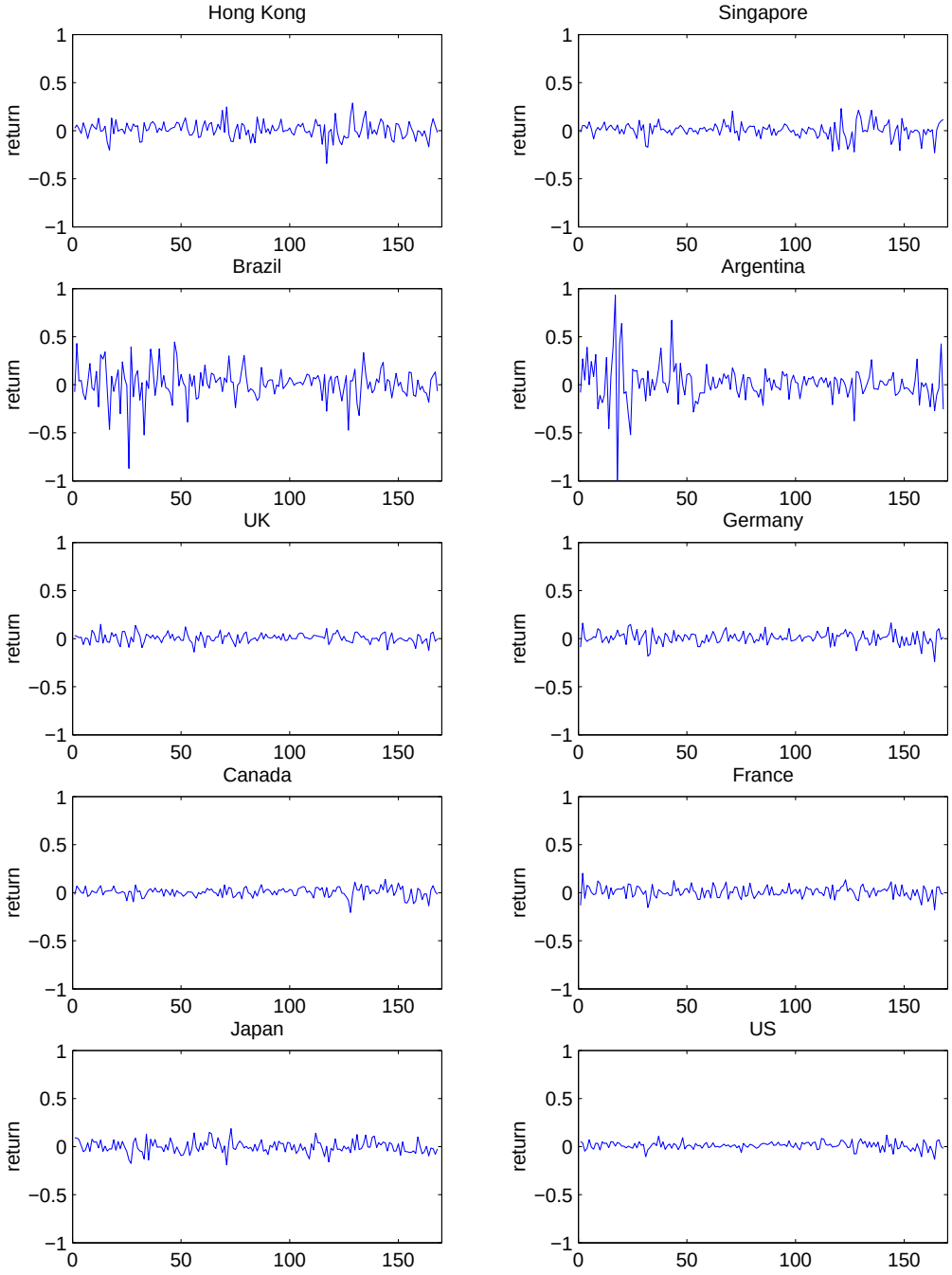

Figure 11.2: *Time series plots of returns for the ten equity markets.*

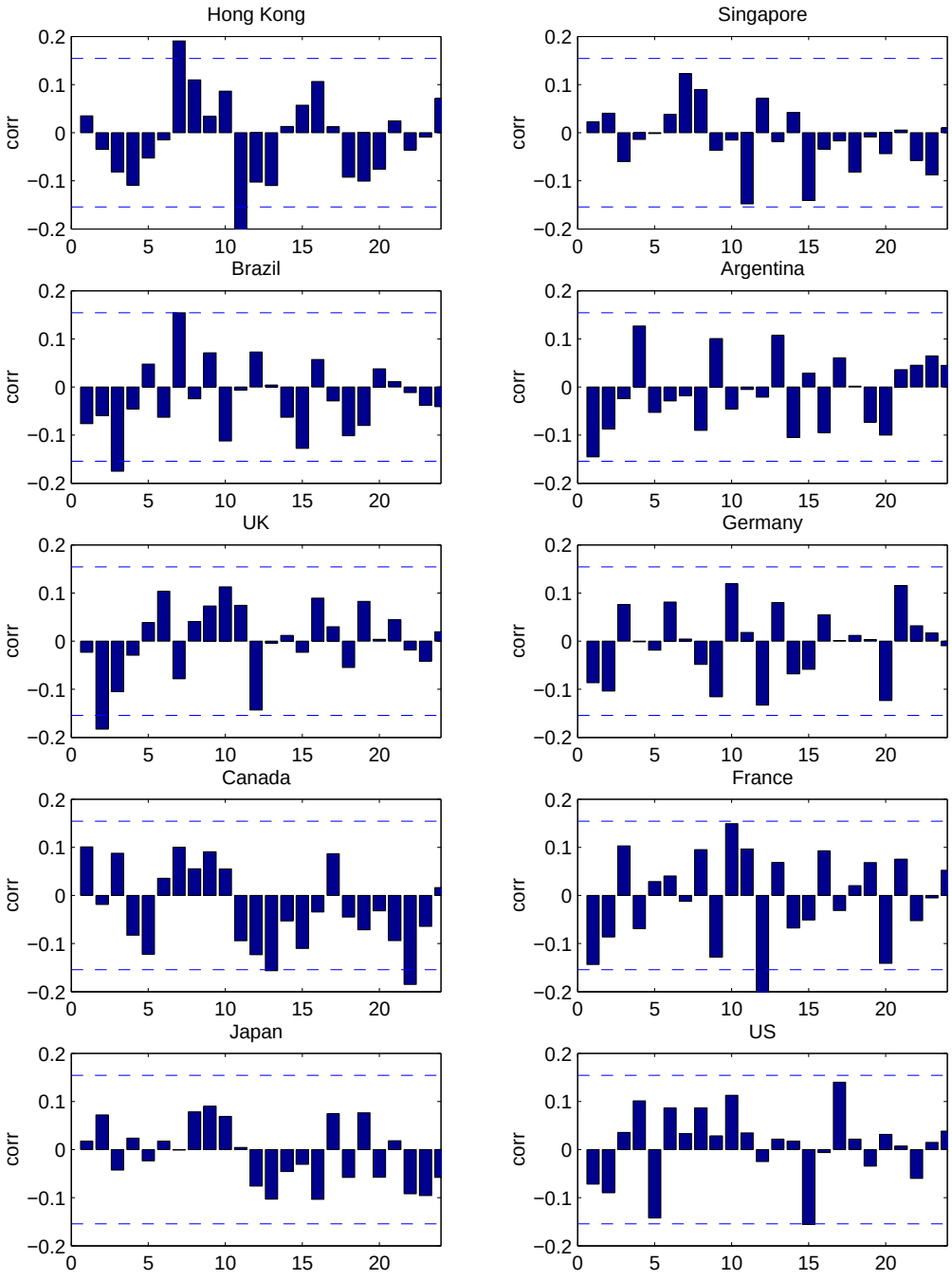

Figure 11.3: *Autocorrelation plots of returns for the ten equity markets.*

Figure 11.3 shows autocorrelation plots of the returns for each of these counries. There is no evidence of nonstationarity though there is some short term autocorrelation.

Figure 11.3 shows normal probability plots of the returns for each of these counries. There is evidence of heavy-tails, especially the left tails.

If we are planning to invest in a number of international capital markets, then we might like to know the efficient frontier, but, of course, we can never know it exactly. At best, we can estimate the efficient frontier using estimated expected returns[4] and the estimated covariance matrix of returns. But how close is the estimated efficient frontier to the unknown true efficient frontier? This question can be addressed by resampling.

Each resample consists of 168 returns drawn with replacement from the 168 returns of the original sample.[5]

From the resampling perspective, the original sample is treated as the population and the efficient frontier calculated from the original sample is viewed as the true efficient frontier. We can repeatedly recalculate the efficient frontier using each of the resamples and compare these re-estimated efficient frontiers to the "true efficient frontier." This is done for six resamples in Figure 11.5. In each subplot, the "true efficient frontier" is shown as a solid blue curve and is labeled "optimal." These curves do not vary from subplot to subplot since then are all calculated from the original sample. The dashed red curves labeled "achieved" are the efficient frontiers calculated from the resamples. "Achieved" means that we calculated the true mean and standard deviation achieved by efficient frontiers calculated from the resamples. "True mean and standard deviation" means we use the original sample (the "population" from the resampling perspective). We can see that there is a fair amount of variability in the resampled efficient frontiers.

[4] In this example, we will use log-returns exclusively, so "return" will always mean "log-return."

[5] There are 169 months of data but we lose one month's data when differencing log prices.

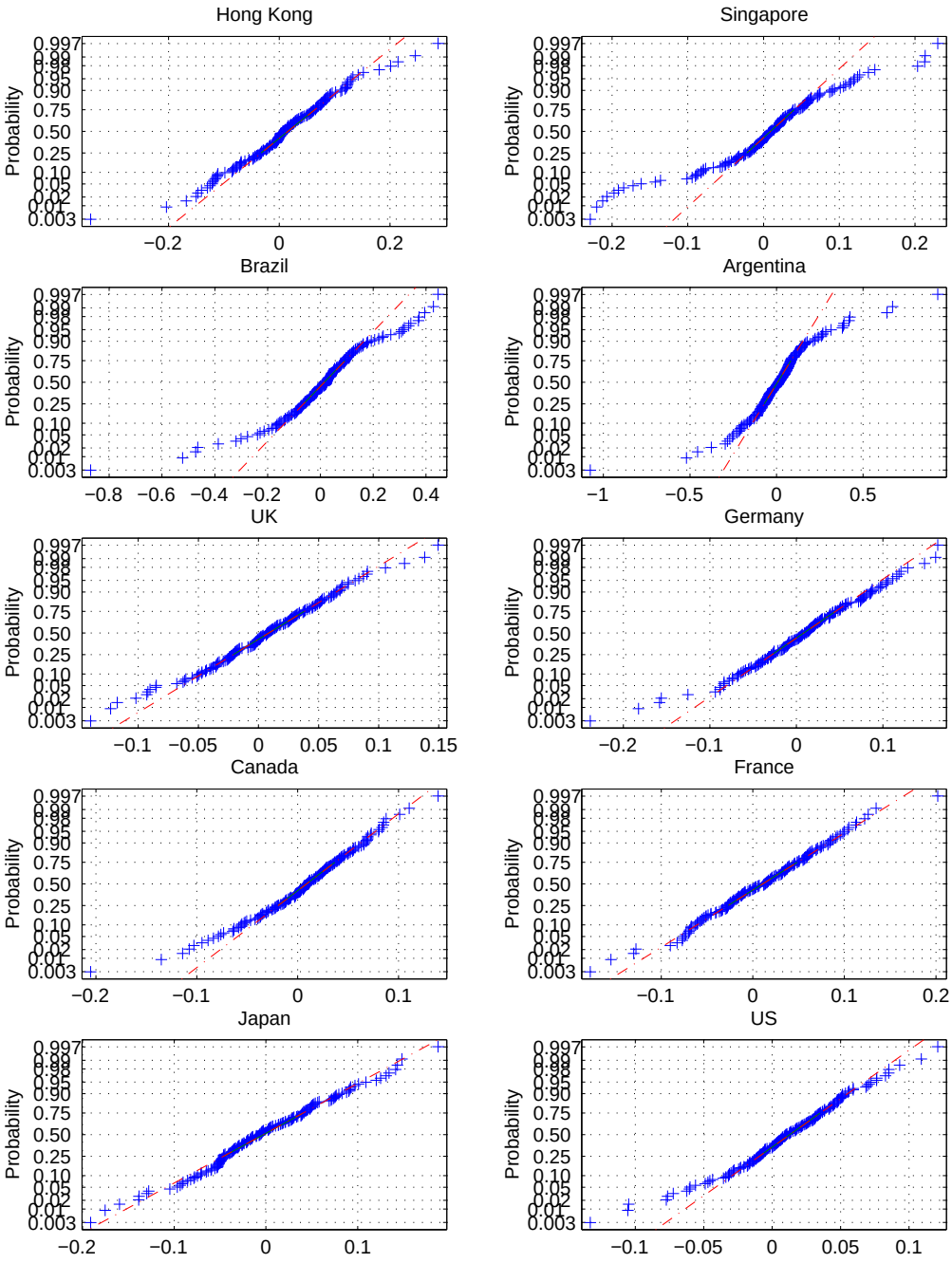

Figure 11.4: *Normal probability plots of returns for the ten equity markets.*

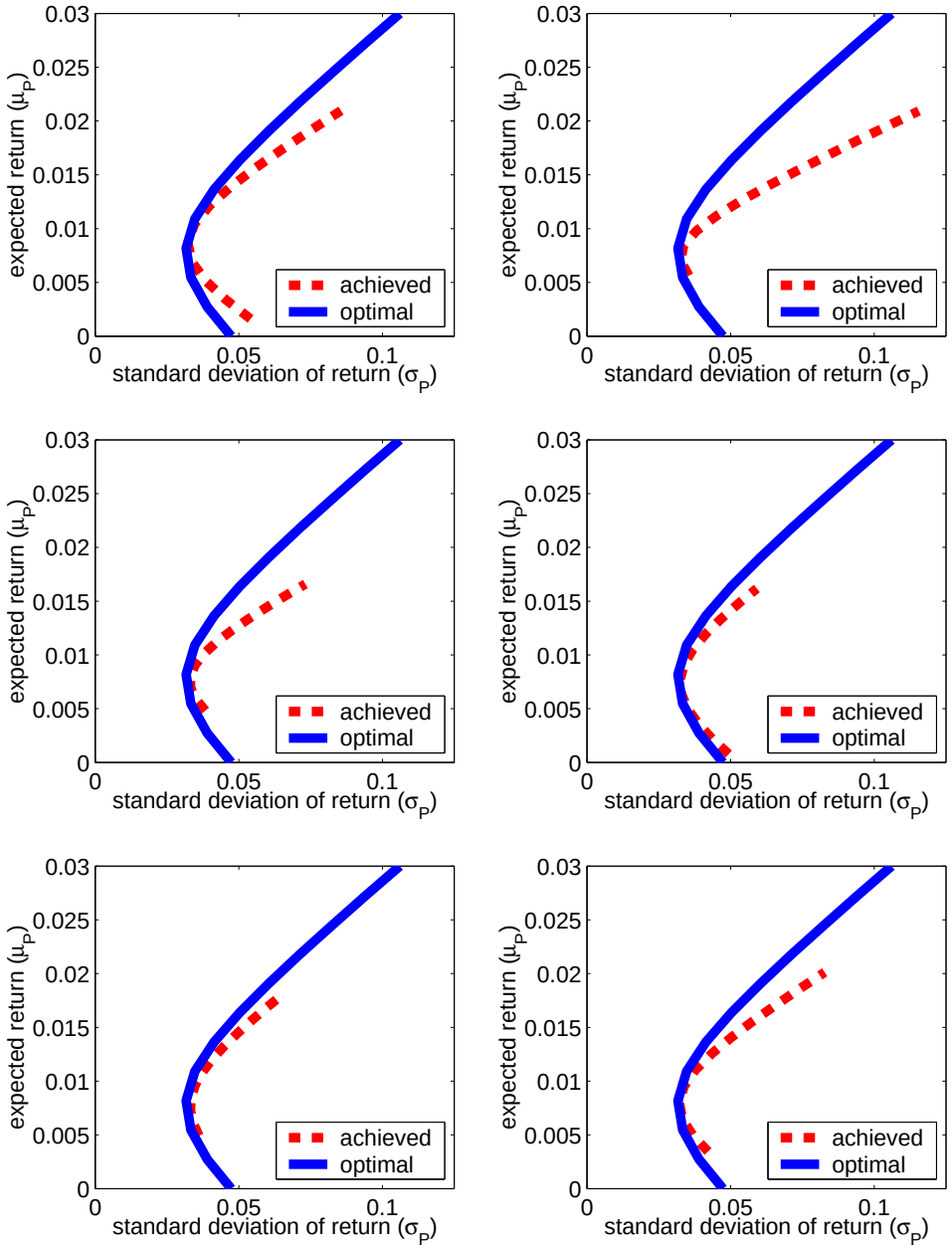

Figure 11.5: *Actual efficient frontier for the sample (optimal) and bootstrap efficient frontier (achieved) for each of six bootstrap resamples.*

To be more precise, let $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Omega}}$ be the mean vector and covariance matrix estimated from the original sample. For a given target for the expected portfolio return, μ_P, let $\widehat{\boldsymbol{\omega}}_{\mu_P}$ be the efficient portfolio weight vector given by equation (5.16) with $\boldsymbol{g}$ and $\boldsymbol{h}$ estimated from the original sample. Let $\widehat{\widehat{\boldsymbol{\omega}}}_{\mu_P,b}$ be the efficient portfolio weight vector with $\boldsymbol{g}$ and $\boldsymbol{h}$ estimated from the bth resample. Then the solid blue curves in Figure 11.5 are $\widehat{\boldsymbol{\omega}}_{\mu_P}^{\mathsf{T}}\widehat{\boldsymbol{\mu}} = \mu_P$ plotted against

$$\sqrt{\widehat{\boldsymbol{\omega}}_{\mu_p}^{\mathsf{T}}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\omega}}_{\mu_P}}$$

for a grid of μ_P values. The dashed red curves are $\widehat{\widehat{\boldsymbol{\omega}}}_{\mu_P,b}^{\mathsf{T}}\widehat{\boldsymbol{\mu}}$ plotted against

$$\sqrt{\widehat{\widehat{\boldsymbol{\omega}}}_{\mu_P,b}^{\mathsf{T}}\widehat{\boldsymbol{\Omega}}\widehat{\widehat{\boldsymbol{\omega}}}_{\mu_P,b}}.$$

Unfortunately, the red resampled efficient frontier curve lies below the blue true efficient frontier curve. Because of estimation error, our estimated efficient frontiers are suboptimal.

Also, $\widehat{\widehat{\boldsymbol{\omega}}}_{\mu_P,b}^{\mathsf{T}}\widehat{\boldsymbol{\mu}}$ does not, in general, equal μ_P. We do not achieve the expected return μ_P that we have targeted because of estimation error when estimating $\boldsymbol{\mu}$. To see this, look at Figure 11.6.

In Figure 11.6, we concentrate on estimating only a single point on the efficient frontier, the point where the expected portfolio return is 0.012. This point is shown in the upper subplot as a large black asterisk and is the point

$$(\sqrt{\widehat{\boldsymbol{\omega}}_{.012}^{\mathsf{T}}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\omega}}_{.012}},\ .012).$$

Each small blue asterisk in the upper subplot is the estimate of this point from a resample and is

$$(\sqrt{\widehat{\widehat{\boldsymbol{\omega}}}_{.012,b}^{\mathsf{T}}\widehat{\boldsymbol{\Omega}}\widehat{\widehat{\boldsymbol{\omega}}}_{.012,b}},\ \widehat{\widehat{\boldsymbol{\omega}}}_{.012,b}^{\mathsf{T}}\widehat{\boldsymbol{\mu}}).$$

The middle subplot of Figure 11.6 is a histogram of the values of

$$\sqrt{\widehat{\widehat{\boldsymbol{\omega}}}_{.012,b}^{\mathsf{T}}\widehat{\boldsymbol{\Omega}}\widehat{\widehat{\boldsymbol{\omega}}}_{.012,b}} - \sqrt{\widehat{\boldsymbol{\omega}}_{.012}^{\mathsf{T}}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\omega}}_{.012}}.$$

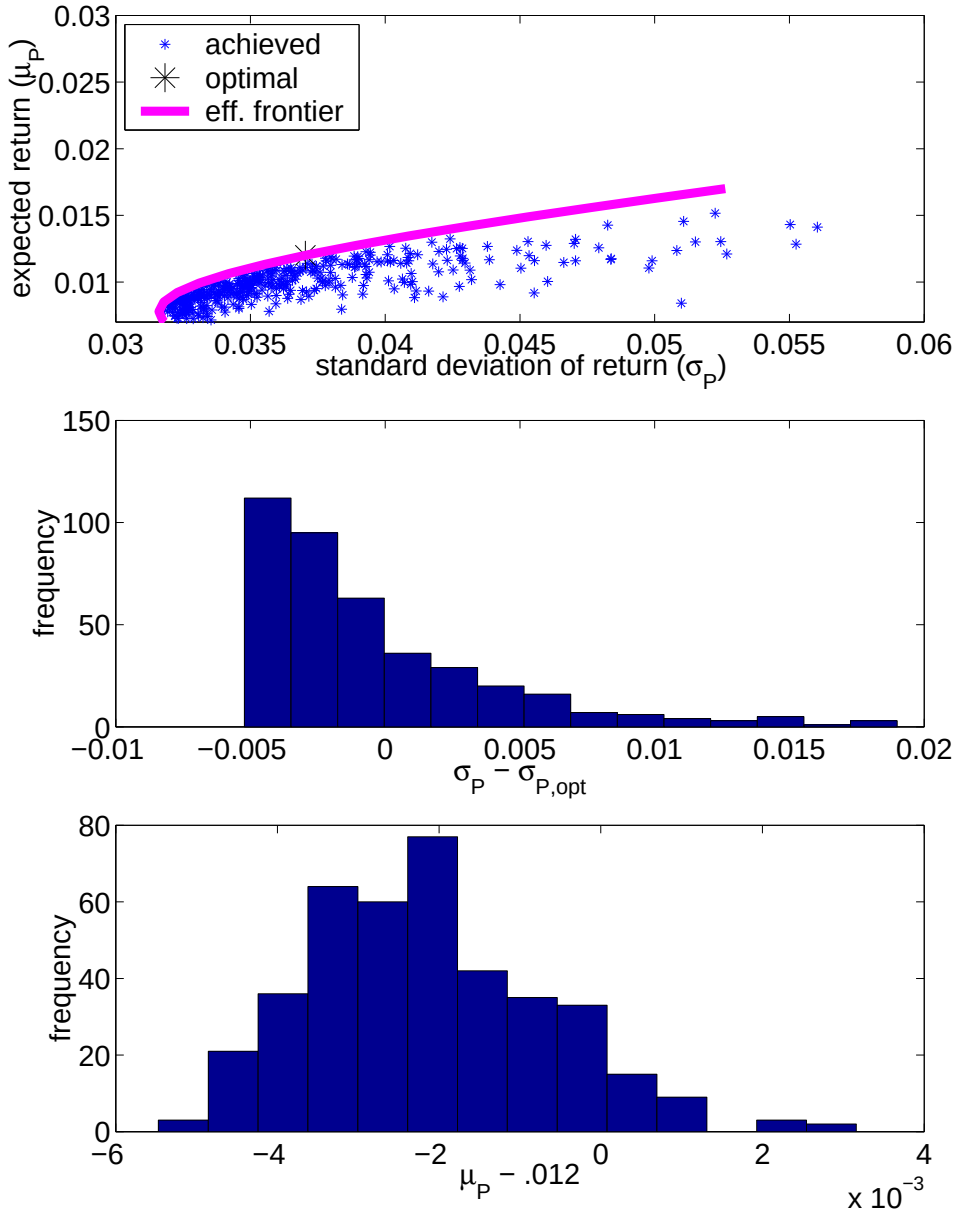

Figure 11.6: *Results from 400 bootstrap resamples. For each resample, the efficient frontier with a mean return of 0.012 is estimated. In the upper subplot, the actual mean return and standard deviation of the return are plotted as a small blue asterisk. The large black asterisk is the point on the efficient frontier with a mean return of 0.012. Histograms of the deviations of σ_P from optimal (middle subplot) and of μ_P from 0.012 (lower subplot) are also shown.*

The lower subplot of Figure 11.6 is a histogram of the values of

$$\widehat{\widehat{\boldsymbol{\omega}}}_{.012,b}^{\mathsf{T}}\widehat{\boldsymbol{\mu}} - .012.$$

An interesting question is "what is the main problem here, mis-estimation of the expected returns, mis-estimation of the covariance matrix of the returns, or both?" One of the fun things we can do with resampling is to play the game of "what if?" In particular, we can ask, "what would happen if we knew the true expected returns and only had to estimate the covariance matrix?" We can also ask the opposite question, "what would happen if we knew the true covariance matrix and only had to estimate the expected returns?" By playing these "what if" games, we can address our question about the relative effects of mis-estimation of $\boldsymbol{\mu}$ and mis-estimation of the covariance matrix.

Figure 11.7 addresses the first of these questions. In that figure, when we estimate the efficient frontier for each resample, we use the mean returns from the original sample, which from the resampling perspective are the population values. Only the covariance matrix is estimated from the resample. Because the true expected returns are known, the estimated efficient portfolio from a resample always does achieve an expected return of .012. The standard deviations of the estimated efficient portfolios are larger than optimal because of estimation error when the covariance matrix is estimated. This can be seen in both the upper subplot and also in the lower subplot which is a histogram of the values of

$$\widehat{\widehat{\boldsymbol{\omega}}}_{.012,b}^{\mathsf{T}}\widehat{\boldsymbol{\mu}} - .012.$$

Figure 11.8 addresses the second of these questions. In this figure, when we estimate the efficient frontier for each resample, we use the covariance matrix from the original sample. Only the expected returns are estimated from the resample. Thus, we see what happens when the "true" covariance matrix is known. Figure 11.8 is surprisingly similar to Figure 11.6. This is an indication that knowing the true covariance matrix does not help us much.

The conclusion from our "what if" games is that the real problem when we estimate the efficient frontier is estimation error in the expected returns.

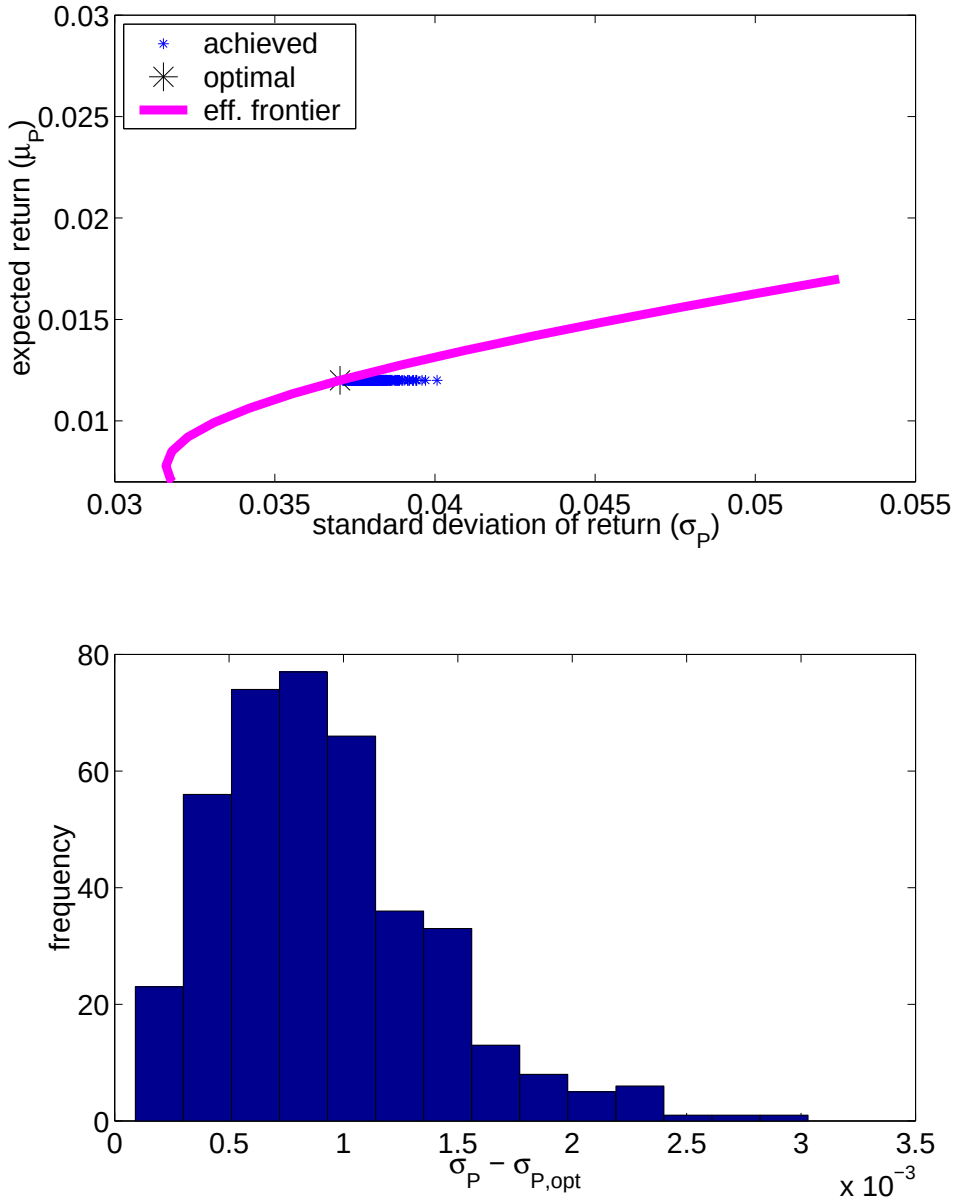

Figure 11.7: *Results from 400 bootstrap resamples assuming that the vector of mean returns is known. For each resample the efficient frontier with a mean return of 0.012 is estimated using the mean returns from the* ***sample*** *and the covariance matrix of returns from the* ***resample****. In the upper subplot, the actual mean returns, which is always 0.012, and standard deviations of the return are plotted as a small blue asterisk. The large black asterisk is the point on the efficient frontier with a mean return of 0.012. A histogram of the deviations of* σ_P *from optimal is also shown (lower subplot).*

This is clearly seen by comparing Figures 11.6 and 11.7. We get very close to the efficient portfolio when the true expected returns are known and only the covariance matrix is estimated.

Here is another "what if" game. This time the question is "what would happen if we had more data?" It is disappointed that the efficient frontier is not estimated more accurately, so it is natural to wonder whether it would be worthwhile to use a longer time series. Without even bothering to collect more data we can assess the effect of more data on accuracy. This assessment is done by using resamples that are *larger* than the sample size n.[6] In Figure 11.9 we repeat the simulation in Figure 11.6 but with the resample size equal to $3n$. Comparing Figures 11.6 and 11.9 we see that more data does help, though perhaps not as much as we would like. Of course, using a longer time series carries the danger that the data may not be stationary. Another is that the MSCI indices may not be available for earlier time periods.

11.4 Summary

- Resampling means taking samples with replacement from the original smaple.
- Resampling is also called the bootstrap.
- Resampling the t-statistic allows us to construct a coefficient interval for the population mean without assuming that the population has a normal distribution.
- Resampling the efficient frontier shows that the efficient frontier is poorly estimated.
- The problem is mainly caused by estimation error in the expected returns.

[6] It was stated earlier that the resample size should equal the sample size. That is true, in general, but here we see that using a resample size different from n is useful if we want to know what would have happened if the sample size had been different.

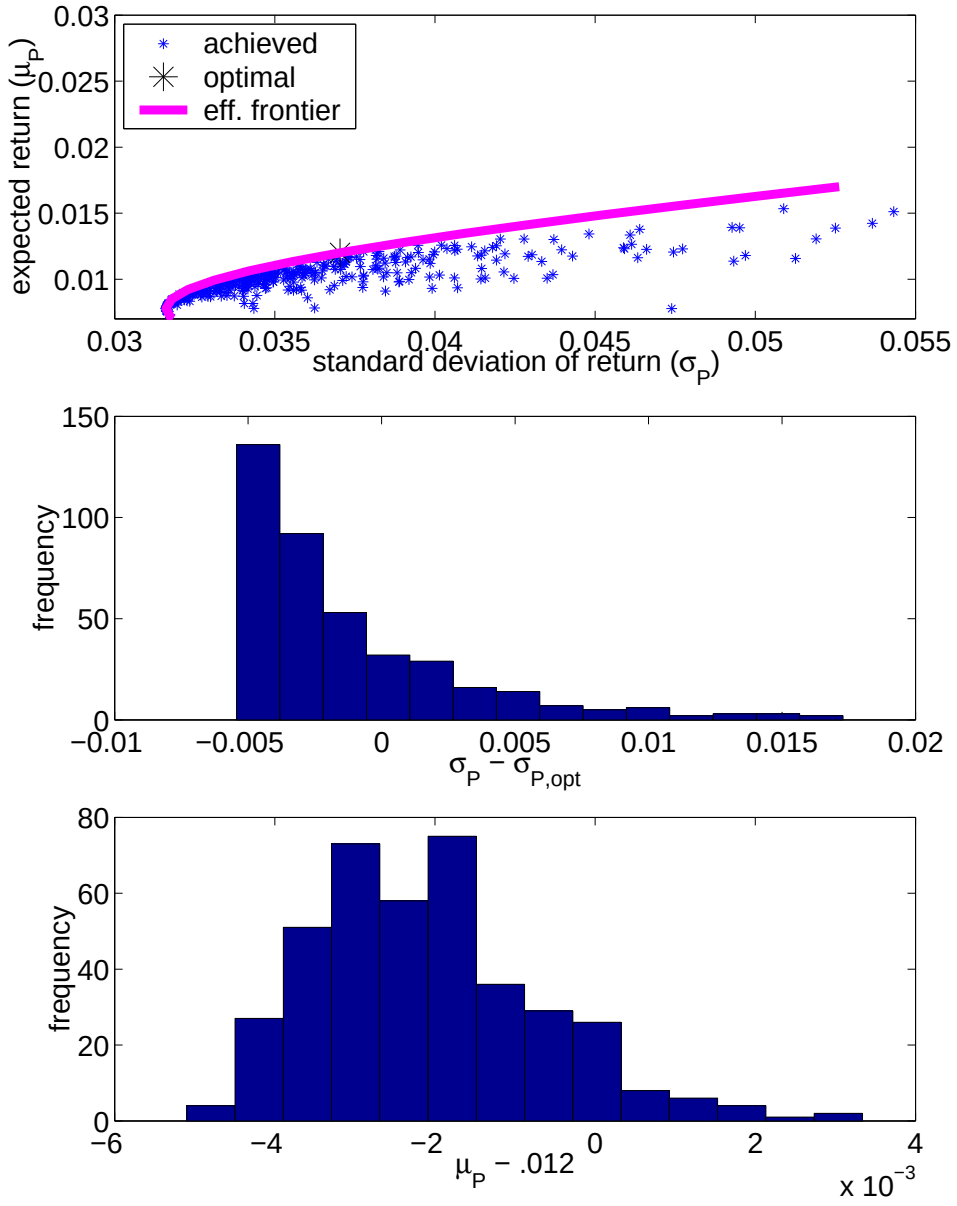

Figure 11.8: *Results from 400 bootstrap resamples assuming that the covariance matrix of the returns is known. For each resample the efficient frontier with a mean return of 0.012 is estimated using the mean returns from the* ***resample*** *and the covariance matrix of returns from the* ***sample****. The actual mean return and standard deviation of the return are plotted as a small blue asterisk. The large black asterisk is the point on the efficient frontier with a mean return of 0.012*

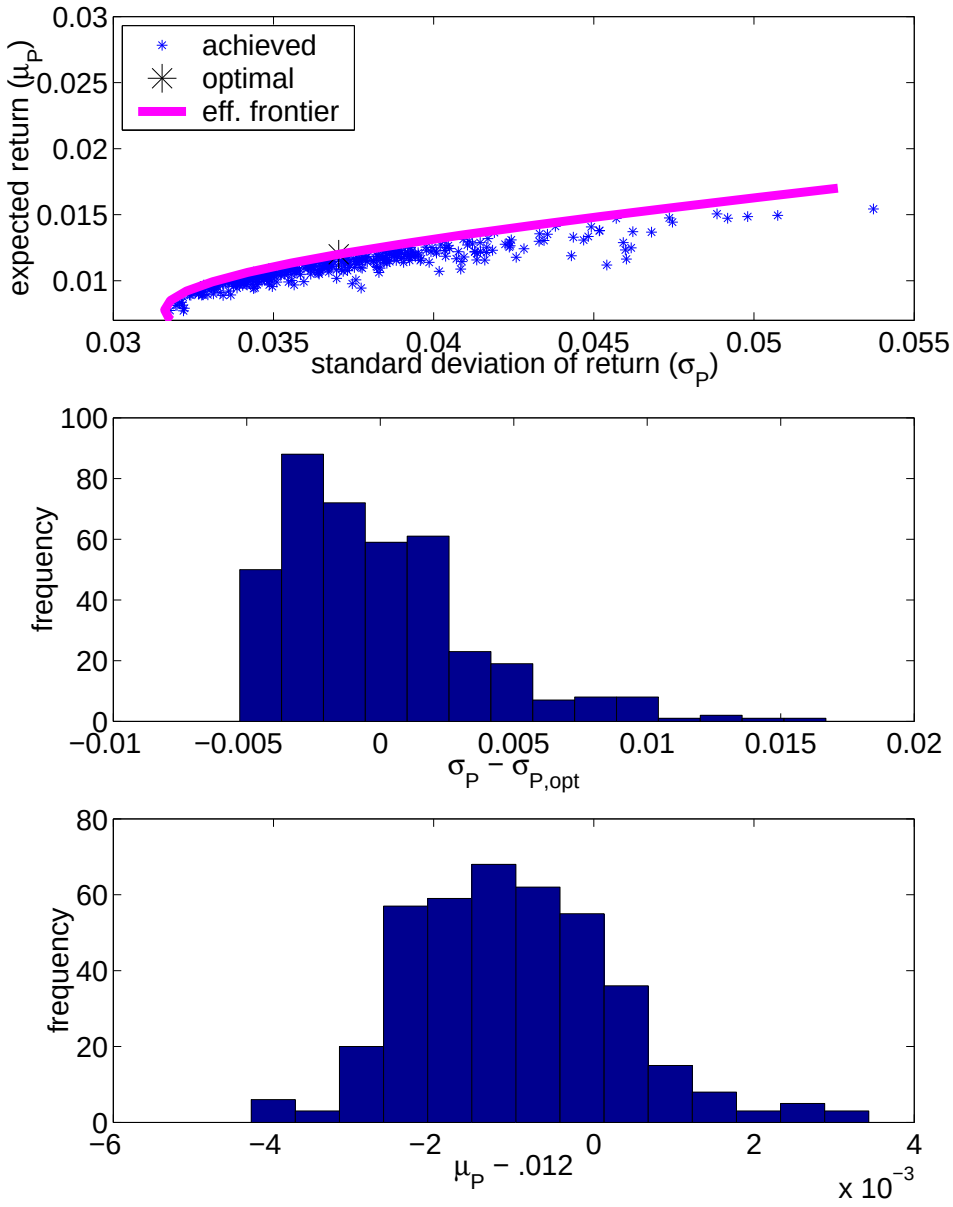

Figure 11.9: *Results from 400 bootstrap resamples with the resample size equal to* $3n$.

11.5 References

Efron, B., and Tibshirani, R., (1993), *An Introduction to the Bootstrap*, Chapman & Hall, New York.

Jobson, J.D., and Korkie, B., (1980), Estimation for Markowitz efficient portfolios, *J. of the American Statistical Association*, 75, 544–554.

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Chapter 12

Splines Models In Finance: 5/1/02

12.1 Introduction

As we have seen in Chapter 6, regression is about modeling the conditional expectation of a response given predictor variables. The conditional expectation is called the regression function. Linear regression assumes that the regression function is a linear function and estimates the intercept and slope, or slopes if there are multiple predictors. Nonlinear regression does not assume linearity but does assume that the regression function is of a known parameter form, for example, an exponential function. In this chapter, we study *nonparametric* regression where the form of the regression function is not specified by a model but rather is estimated from data. Nonparametric regression is used when we know or suspect that the regression function is curved, but we do not have a model for the curve.

There are many techniques for nonparametric regression, but in my experience splines are the easiest to use and as well as the easiest to understand because they are a natural extension of linear regression. As mentined in Section 10.8, a spline is a function constructed by piecing together polynomial functions. The spline modeling techniques studied in the chapter can be used in a wide variety of practical problems including modeling financial markets data.

Models for evolution of short-term interest rates are important in fi-

nance, for example, because they are need for the pricing of interest rate derivatives. In this chapter, we will use an example of short-term Euro rates introduced by Yau and Kohn (2001), two statisticians at the Australian Graduate School of Management in Sydney. Figure 12.1 shows data on the interest rate of Euro dollar deposits of one-month maturity. The top plot is the weekly time series of the Euro interest rate. The bottom plot is the time series of weekly changes in the Euro rate which were found by differencing the original series. What interests us here is how the volatility of the changes in the interest rate depends on the current value of the rate. To address this question, Figure 12.2 shows the differences and squared differences of the interest rate plotted again the rate itself, that is, $(r_{t+1} - r_t)$ and $(r_{t+1} - r_t)^2$ plotted against r_t where r_t is the interest rate at time t.

A common model for changes in short-term interest rates is

$$\Delta r_t = \mu(r_{t-1}) + \sigma(r_{t-1})\epsilon_t, \tag{12.1}$$

where $\Delta r_t = r_t - r_{t-1}$, $\mu(\cdot)$ is the drift function, $\sigma(\cdot)$ is the volatility function, also called the diffusion function, and ϵ_t is $N(0,1)$ noise. As discussed in Yau and Kohn (2001), many different parametric models have been proposed fro $\mu(\cdot)$ and $\sigma(\cdot)$, for example, by Merton (1973), Vasicek (1977), Cox, Ingersoll, and Ross (1985), and Chan et al. (1992). The simplest model, the one due to Merton (1973), is that $\mu(\cdot)$ and $\sigma(\cdot)$ are constant. Chan et al. (1992) assume that $\mu(r) = \beta(r - \alpha)$ and $\sigma(r) = \theta r^\gamma$ where α, β, θ, and γ are unknown parameters. The approach of Yau and Kohn (2001) that is followed here is to model both $\mu(\cdot)$ and $\sigma(\cdot)$ nonparametrically. Doing this allows one to check which parametric models fit the data, if any, and to have a nonparametric alternative if none of the parametric models fits well.

The solid red curves Figure 12.2 in are estimates of $\mu(\cdot)$ and $\sigma^2(\cdot)$ based on a nonparametric regression method called *penalized splines*.[1] The estimate of $\mu(\cdot)$ seems to be zero or nearly so. It will be assumed that $\mu(\cdot)$ is 0, in which case

$$E((\Delta r_t)^2 | r_{t-1}) = \sigma^2(r_{t-1}).$$

Therefore, $\sigma^2(\cdot)$ is estimated by regressing $(\Delta r_t)^2$ on r_{t-1}.

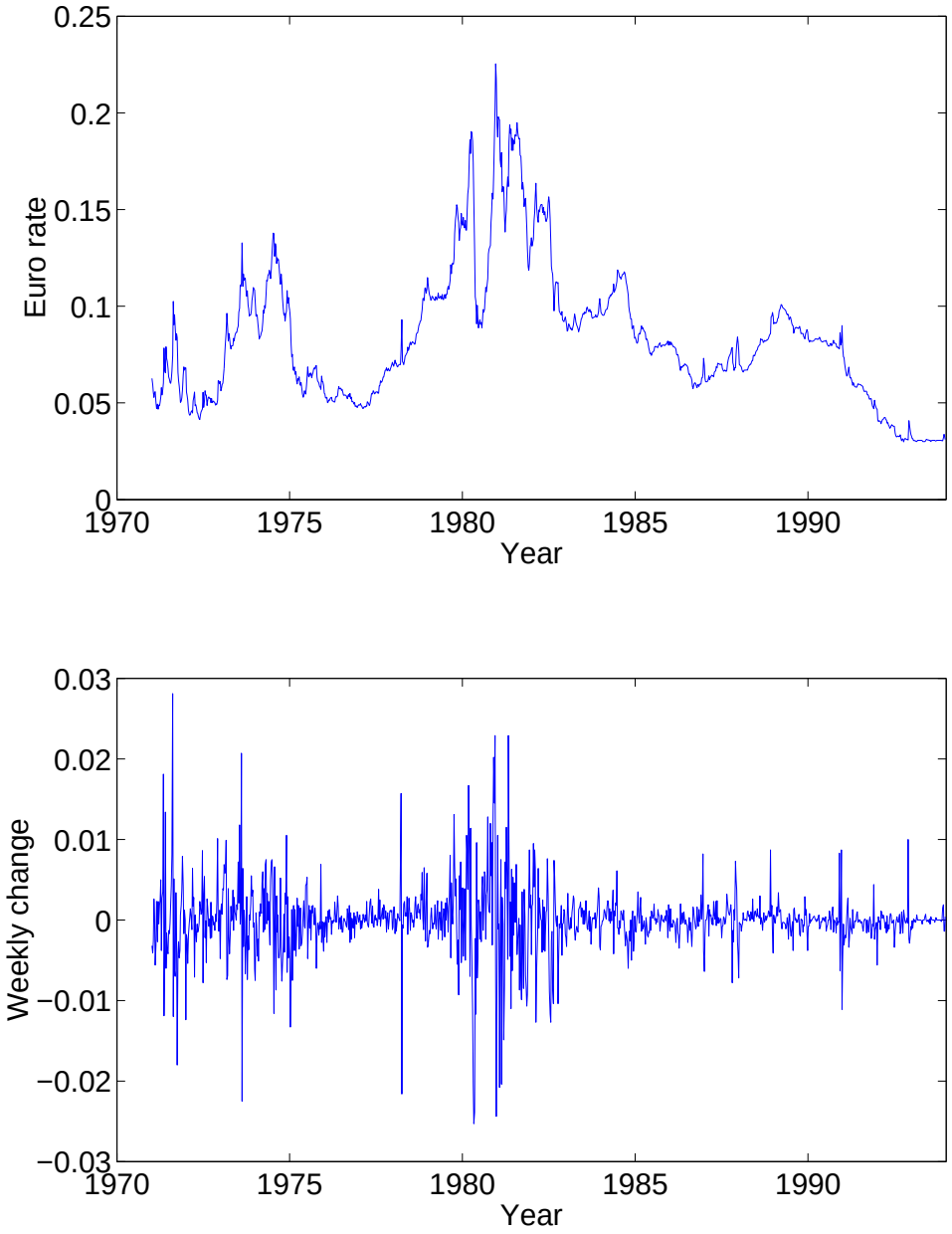

Figure 12.1: *Study of volatility in weekly changes of one-month Euro bond rate. The top plot is the time series of weekly values of the interest rate. The bottom plot is the time series of changes in the interest rate.*

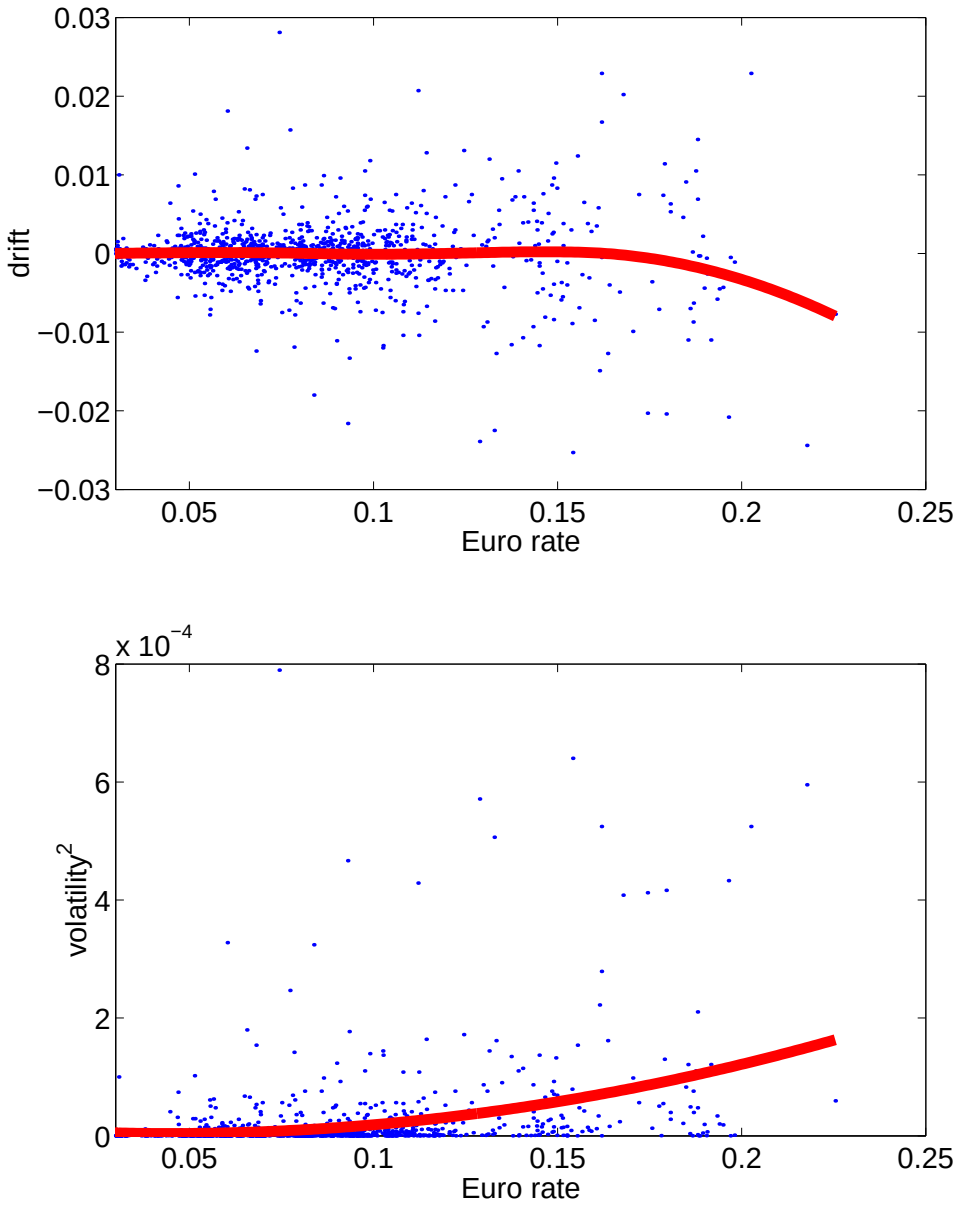

Figure 12.2: *Study of volatility in weekly changes of Euro bond rate. The top plot is a plot of weekly rate changes against the rate itself. The solid red curve is a 20-knot P-spline estimate of $\mu(\cdot)$. The bottom plot is a plot of squared weekly rate changes against the rate. The solid red curve is a 20-knot P-spline estimate of $\sigma(\cdot)$. The penalty parameter of both P-splines was selected by GCV.*

12.2 Guidelines for using nonparametric regression

Here are some guidelines to help decide when to use linear, nonlinear, or nonparametric regression.

Nonparametric Nonparametric regression can be used when there is no theory to suggest the functional form of the regression function.

Nonparametric regression is most useful when there is enough information in the data that the form of the regression function can be recovered. In particular, it is desirable that:

1. there are enough data points,
2. the noise in the data is sufficiently small, or
3. the predictor variable(s) vary over a wide enough range

that nonlinearities in the regression function can be detected. When none of conditions 1–3 are not met, then nonparametric is not necessarily inappropriate but may not be much different than using linear regression.

When one has little idea about the shape of the regression function, then nonparametric regression is a good place to start. If the regression function really is linear, then the linearity should be apparent in the nonparametric fit.

Linear Linear regression is often used when there is no theory to suggest the shape of regression function and the data are not very informative about the shape either. The data can be noninformative about the shape of the regression function if the data are very noisy, there are too few data, and the predictor variable(s) vary too little.

More rarely, linear regression is used because there is a theoretical model that suggests that the regression function will, in fact, be linear.

[1] Yau and Kohn (2001) also use spline-based estimates though not penalized splines.

Nonlinear regression Nonlinear regression is used when there is a theory that suggests a specific nonlinear model, for example, equation (10.26) for the price of a zero coupon bond.

Sometimes a nonlinear model is used, not because it was derived from theory, but because experience has shown that the model generally fits data of a particular type.

Nonlinear regression is also used when the range of the response is restricted, for example, when the response is a proportion so that it lies between 0 and 1. In such cases, one can use a nonlinear regression function whose range is the same as the range of the response.

Figures 12.3 to 12.7 use simulated data to illustrate when linear regression is appropriate and when it is preferable to use nonparametric regression. In all five figures the regression function is nonlinear, $2x^2 \sin(4x)$, but it is assumed that this functional form of the curve is unknown to the data analyst. The figures differ in their values of σ, n, and the range of x. In each figure, a fit by linear regression is shown along with a nonparametric fit using a quadratic P-spline (penalized spline). P-splines are introduced in Section 12.7.

In Figure 12.3, $\sigma = .4$, $n = 15$, and the range of x is [0, .7]. With this relatively large value of σ, small sample size, and short range of x, the nonlinearity of the regression function is difficult to detect. The linear fit and the nonparametric fit are similar and neither shows any of the curvature seen in the true regression function. Although in this example nonparametric regression cannot successfully recover the curvature in the regression function, using nonparametric regression is not inappropriate here. Nonparametric regression does as well as possible with this data set and gives essentially the same answer as linear regression so one cannot say that using linear regression would be a much better alternative.

In each of the next three figures, we vary one of the parameters in Figure 12.3 so that either

1. the noise σ is smaller,
2. the range of x is larger, or
3. the sample size n is larger.

Any one of these changes allows us to detect the curvature in the regression function.

Figure 12.4 is similar to Figure 12.3 except that σ is much smaller, only .05, in Figure 12.4. Because of the low noise, the nonlinearity in the regression can be detected in Figure 12.4 and compared to the linear fit, the nonparametric fit is much closer to the true regression function. Notice that the nonparametric fit comes quite close to the true regression function without having an prior knowledge of the shape of that curve.

Figure 12.5 is similar to Figure 12.3 except that the range of x is larger in Figure 12.5 so that the nonlinearity in the regression function can be detected.

In Figure 12.6, the range of x is short and σ is large, but the nonlinearity can be detected because the sample size is large, $n = 300$.

Bear in mind that all differentiable functions look like straight lines over very short ranges of x. For example, Figure 12.7 is similar to Figure 12.4 except the range of x is very short, [0, .2]. Within this short range, the regression function is relatively linear. Even though σ here is the same small value as in Figure 12.4, the deviation from linearity is no longer obvious. Therefore, in Figure 12.7 the linear and nonparametric fits are similar.

Although the nonparametric fit in Figure 12.7 shows some curvature, the curvature is due to the nonparametric fit following the noise in the data, not the signal. For example, by chance the three leftmost data point in Figure 12.4 all lie below the true curve. The nonparametric fit dips down to follow these data points and misses the true curve by a substantial amount. The linear fit cannot dip down like this and on the left side of the plot, compared to the nonparametric fit, the linear fit is closer to the true curve.

12.2.1 Summary

The bottom line is that it never hurts to look a a nonparametric regression estimate and to compare this estimate with a parametric estimate.

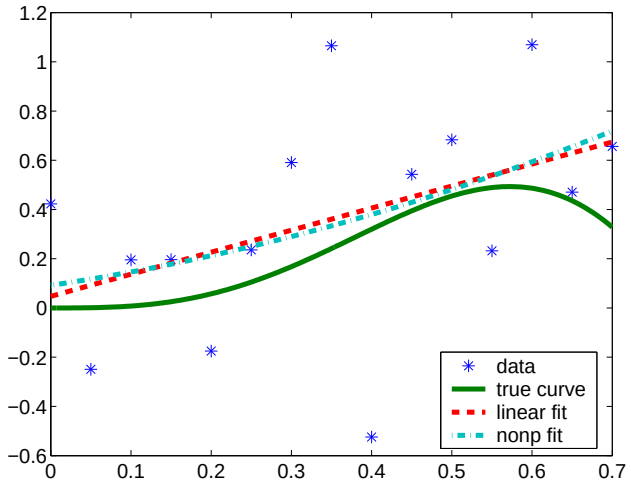

Figure 12.3: *Simulated regression example with regression function* $2x^2 \sin(4x)$. *Here* $\sigma = .4$, $n = 15$, *and the range of* x *is* $[0, .7]$. *In this example,* σ *is relatively large,* n *is small, and the range of* x *is relatively small. Therefore, the nonlinearity in the regression function can not be detected from the data and the nonparametric and linear regression fits are similar.*

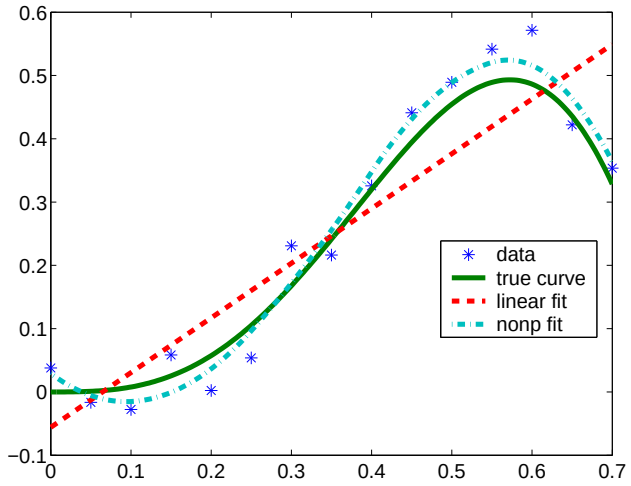

Figure 12.4: *Simulated regression example with regression function* $2x^2 \sin(4x)$. *Here* $\sigma = .05$, $n = 15$, *and the range of* x *is* $[0, .7]$. *Because of the low noise, the nonlinearity of the regression function is evident and the nonparametric fit is close to the true curve.*

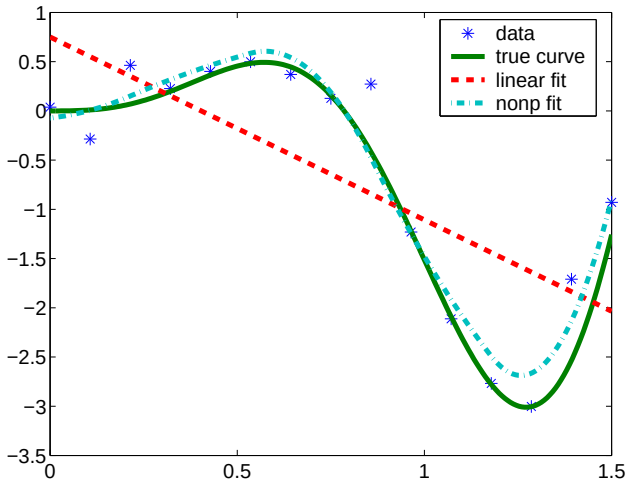

Figure 12.5: *Simulated regression example with regression function* $2x^2 \sin(4x)$. *Here* $\sigma = .4$, $n = 15$, *and the range of* x *is* $[0, 1.5]$. *Because of the wide range of* x, *the nonlinearity is seen in the data and captured by the nonparametric fit.*

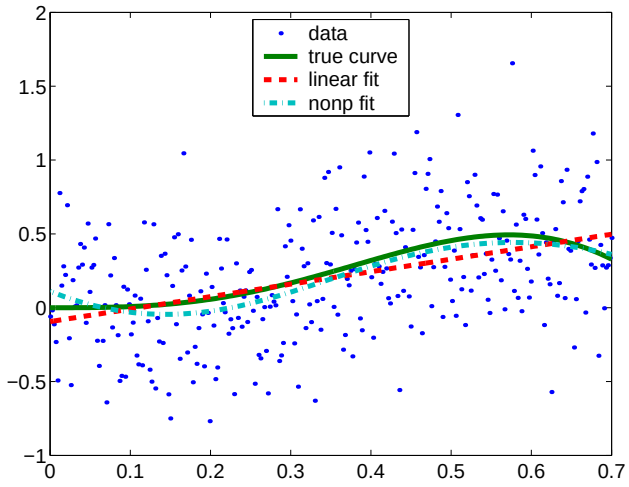

Figure 12.6: *Simulated regression example with regression function* $2x^2 \sin(4x)$. *Here* $\sigma = .4$, $n = 300$, *and the range of* x *is* $[0, .7]$. *Because of the large sample size, the nonlinearity in the regression function can be detected.*

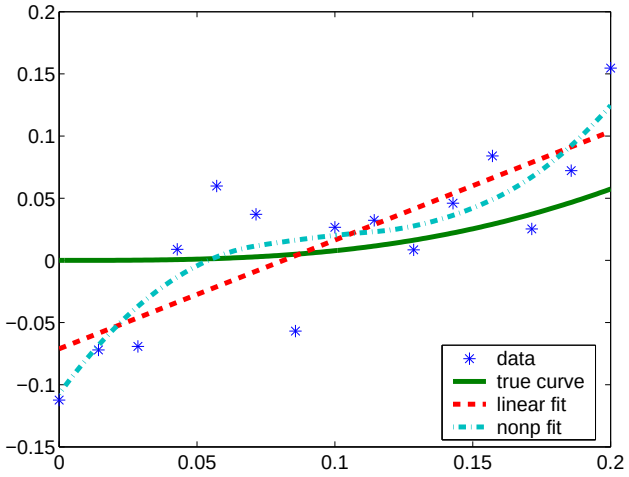

Figure 12.7: *Simulated regression example with regression function* $2x^2 \sin(4x)$. *Here* $\sigma = .05$, $n = 15$, *and the range of* x *is* $[0, .2]$.

12.3 Linear splines

A linear splice is constructed by piecing together linear functions so that they join together at specified locations called "knots."

12.3.1 Linear splines with one knot

We will start simple, a linear spline with one knot. Figure 12.8 illustrates such a spline. This spline is defined as

$$\begin{aligned} f(x) &= .5 + .2x, & x < 2, \\ &= -.5 + .7x, & x \geq 2. \end{aligned}$$

Because $.5 + (.2)(2) = .9 = -.5 + (.7)(2)$, the two linear components are equal at the point $x = 2$ so that they join together there.

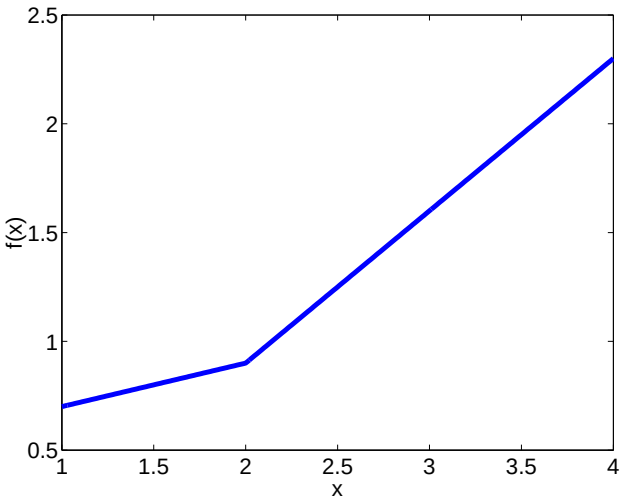

Figure 12.8: *Example of a linear spline with a knot at 2.*

The point $x = 2$ where the spline switches from one linear function to the other is called a *knot*. A linear spline with a knot at the point t can be constructed as follows. The spline is defined to be $s(x) = a + bx$ for $x < t$

and $s(x) = c + dx$ for $x > t$. The parameters a, b, c, and d can be chosen arbitrarily except that they must satisfy the equality constraint

$$a + bt = c + dt, \tag{12.2}$$

which assures us that the two lines join together at t. Solving for c in (12.2), we get $c = a + (b - d)t$. Substituting this expression for c into the definition of $s(x)$ and doing some rearranging, we have

$$\begin{array}{lll} s(x) & = a + bx, & x < t, \\ & = a + bx + (d - b)(x - t), & x > t. \end{array}$$

Recall the definition that for any number y

$$\begin{array}{lll} (y)_+ & = 0, & y < 0, \\ & = y, & y > 0. \end{array}$$

By this definition

$$\begin{array}{lll} (x - t)_+ & = 0, & x < t, \\ & = (x - t), & x > t. \end{array}$$

We will call $(x - t)_+$ a linear *plus function* with knot at t. The spline s can be written using this plus function:

$$s(x) = a + bx + (d - b)(x - t)_+.$$

In summary, if we want a linear spline that equals $a + bx$ for $x < t$ and then has its slope jump from b to d at $x = t$, then the spline is $a + bx$ plus the jump, $(d - b)$, times the plus function $(x - t)_+$.

Figure 12.9 illustrates a linear plus function with a knot at 1 and its first derivative. Notice that

$$\begin{array}{lll} \frac{d}{dx}(x - t)_+ & = 0, & x < t, \\ & = 1, & x > t. \end{array}$$

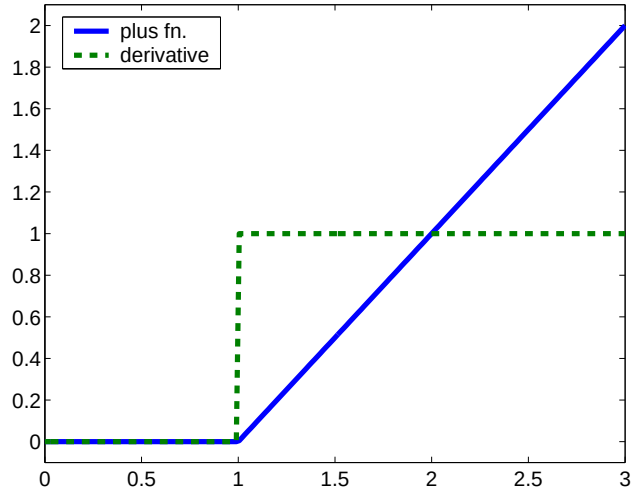

Figure 12.9: *The linear plus function* $(x-1)_+$ *with knot at 1 and its first derivative.*

12.3.2 Linear splines with many knots

Plus functions are very convenient when defining splines with more than one knot because plus functions automatically join the component linear functions together so that the spline is continuous. For example, suppose we want a linear spline to have K knots, $t_1 < \cdots < t_K$, for the spline to equal $s(x) = \beta_0 + \beta_1 x$ for $x < t_1$, and for the first derivative of the spline to jump by the amount b_k at knot t_k, for $k = 1, \ldots, K$. Then the spline can be constructed from linear plus functions, one for each knot:

$$s(x) = \beta_0 + \beta_1 x + b_1(x - t_1)_+ + b_2(x - t_2)_+ + \cdots + b_K(x - t_K)_+.$$

Because the plus functions are continuous, the spline is the sum of continuous functions and is therefore continuous itself—the continuity of the spline is automatically "inherited" from the plus functions.

12.4 Other degree splines

12.4.1 Quadratic splines

A linear spline is continuous but has "kinks" at its knots where its first derivative jumps. If we want a function without these kinks, we cannot

use a linear spline. A quadratic spline is a function obtained by piecing together quadratic polynomials. In other words, $s(x)$ is a quadratic spline with knots $t_1 < \cdots < t_K$ if $s(x)$ equals one quadratic polynomial to the left of t_1 and equals a second quadratic polynomial between t_1 and t_2, etc. The quadratic polynomials are pieced together so that the spline is continuous and, to guarantee no kinks, its first derivative is also continuous.

As with linear splines, continuity can be enforced by using plus functions. Define the quadratic plus function

$$\begin{aligned}(x-t)_+^2 &= 0, && x < t\\ &= (x-t)^2, && x > t.\end{aligned}$$

Notice that $(x-t)_+^2$ equals $\{(x-t)_+\}^2$, not $\{(x-t)^2\}_+ = (x-t)^2$.

Figure 12.10 shows a quadratic plus function and its first and second derivative. One can see that

$$\frac{d}{dx}(x-t)_+^2 = 2(x-t)_+$$

and

$$\frac{d^2}{dx^2}(x-t)_+^2 = 2(x-t)_+^0$$

where $(x-t)_+^0 = \{(x-t)_+\}^0$ so that

$$\begin{aligned}(x-t)_+^0 &= 1 && x > t,\\ &= 0 && x < t.\end{aligned}$$

Therefore, the second derivative of $(x-t)_+^2$ jumps from 0 to 2 at the knot t.

A quadratic spline with knots $t_1 < \cdots < t_K$ can be written as

$$s(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + b_1(x-t_1)_+^2 + b_2(x-t_2)_+^2 + \cdots + b_K(x-t_K)_+^2.$$

The second derivative of s jumps by the amount $2b_k$ at knot t_k for $k = 1, \ldots, K$.

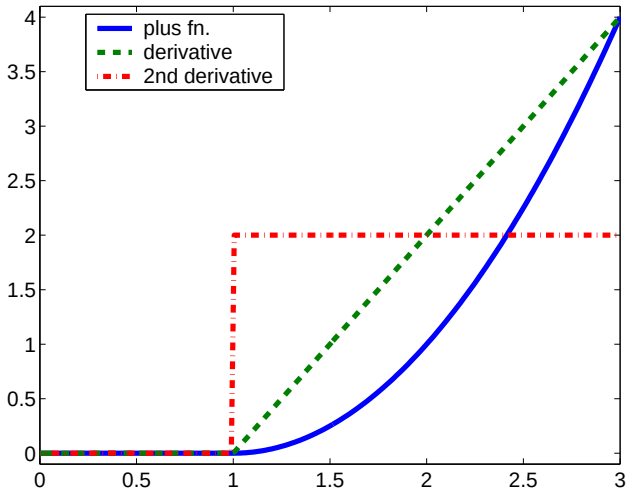

Figure 12.10: *The quadratic plus function* $(x-1)_+^2$ *with knot at 1 and its first and second derivatives.*

12.4.2 pth degree splines

The way to define a pth degree spline with knots $t_1 < \cdots < t_K$ should now be obvious:

$$s(x) = \beta_0 + \beta_1 x + \cdots + \beta_p x^p + b_1(x-t_1)_+^p + \cdots + b_K(x-t_K)_+^p, \quad (12.3)$$

where as we have seen for the specific case of $p = 2$, $(x-t)_+^p$ equal $\{(x-t)_+\}^p$.

The first $p-1$ derivatives of s are continuous while the pth derivative takes a jump equal to $p!\, b_k$ at the kth knot.

It has been my experience that linear and quadratic splines work well in practice and I do not see a need to use cubic or higher degree splines except for some special purposes.

One application where higher order splines are useful is when one wants to estimate not the regression function itself, but rather one of the derivatives of the regression function. The derivative of a pth degree spline is a spline of degree $p-1$. So, for example, if one starts with a linear spline and differentiates it, then one obtains a 0-degree spline, which may not be desirable. A 0-degree spline is a constant (0-degree polynomial) between its knots and takes jumps at the knots. 0-degree splines are often called step

functions or piecewise constant functions.

12.5 Least-squares estimation

A pth spline with knots $t_1 < \cdots < t_K$ can be easily fit to data by least-squares. One simply sets up a multiple regression model with predictor variables $x, \ldots, x^p, (x - t_1)_+^p, \ldots, (x - t_K)_+^p$. The regression model is

$$Y_i = \beta_0 + \beta_1 X_i + \cdots + \beta_p X_i^p + b_1 (X_i - t_1)_+^p + \cdots + b_K (X_i - t_K)_+^P + \epsilon_i \quad (12.4)$$

where the regression function is the function s given by equation (12.3).

Here is a MATLAB program to fit a linear spline to the data in the bottom of Figure 12.2.

```
predictors = [rate   (rate - .08).*(rate > .08) ...
  (rate - .12).*(rate > .12)  ...
  (rate - .16).*(rate > .16) ...
  (rate - .2).*(rate > .2) ] ;
fit1 = linregress(predictors,diff) ;
fit2 = linregress(predictors, (diff - fit1.yhat).^2 ) ;
rate_grid = linspace(min(rate),max(rate),1000)' ;
predictors_grid = [ones(1000,1) rate_grid   ...
          (rate_grid - .08).*(rate_grid > .08) ...
  (rate_grid - .12).*(rate_grid > .12)  ...
  (rate_grid - .16).*(rate_grid > .16) ...
  (rate_grid - .2).*(rate_grid > .2) ] ;
vol_est = predictors_grid*fit2.beta ;
```

The spline has knots at .08, .12, .16, and .2. In this program "rate" and "diff" are the Euro rate and the first difference of the Euro rate, respectively. In MATLAB the variable "(rate $>$ .08)" is a logical variable that equals 1 if "rate" is greater than .08 and equals 0 otherwise. Therefore, (rate $-$ .08).*(rate $>$ .08) is a linear plus function with knot at .08. The matrix "predictors" has five columns. The first column contains "rate," the second column contains the plus function (rate $-$ .08).*(rate $>$ .08) and columns three to five contain the other plus functions.

This program calls the linear regression function "linregress" with "predictors" as the matrix of predictor variables and the response equal to the "diff" squared. "linregress" returns a MATLAB *structure* that is named "fit." This structure contains the output of the regression.

A structure in MATLAB is an object containing variables. By grouping variables into a structure we can refer to all of the variables using the structure's name. The individual elements of the structure are easy to access. For example, in this program, the fitted values from the regression are referred to as "fit.yhat."

The general call to linregress is:

```
structurename = linregress(predictorname,responsename) ;
```

See the comments in the program "linregress" for a complete description of the output. Many of the statistics returned by this program are advanced regression diagnostics that are discussed in courses on regression but will not be used in this course.

"rate_grid" is a 1000-point grid from the smallest to the largest observed values of "rate." "vol_est" is the regression function estimate on this grid. The regression function was computed on a fine grid in order to show fine detail in the estimate, especially the kinks at the knots.

Figure 12.11 is the plot produced by this program, specifically by the lines:

```
p1 = plot(rate,(diff-fit1.yhat).^2,'.',rate_grid,vol_est) ;
set(p1(2),'linewidth',8) ;
set(p1(2),'color','red') ;
set(gca,'fontsize',16) ;
xlabel('Euro rate','fontsize',16) ;
ylabel('volatility^2','fontsize',16) ;
```

The first line attaches the *handle* "p1" to the plot so that attributes of the plot can be changed in the following lines.

The kink at .2 is evident, but the kinks at .08, .12, and .16 are less visible.

To remove the kinks in the estimated regression function, we can fit a quadratic spline to the data. The only change in the above program is the change in the definition of the matrix "predictors." The square of the variable "rate" is added to this matrix and the plus functions are changed with (rate $-$ .08).*(rate $>$ .08) being replaced by (rate $-$ .08).^2.*(rate $>$ .08), etc. Here is the code that has been changed:

```
predictors = [rate  rate.^2 ...
 (rate - .08).^2.*(rate > .08)  ...
 (rate - .12).^2.*(rate > .12)  ...
 (rate - .16).^2.*(rate > .16)  ...
 (rate - .2).^2.*(rate > .2)  ] ;
```

There is a similar change to "predictors_grid." Figure 12.12 is the plot produced by the new program. The "kink" at .2 is replaced by a smoother curve.

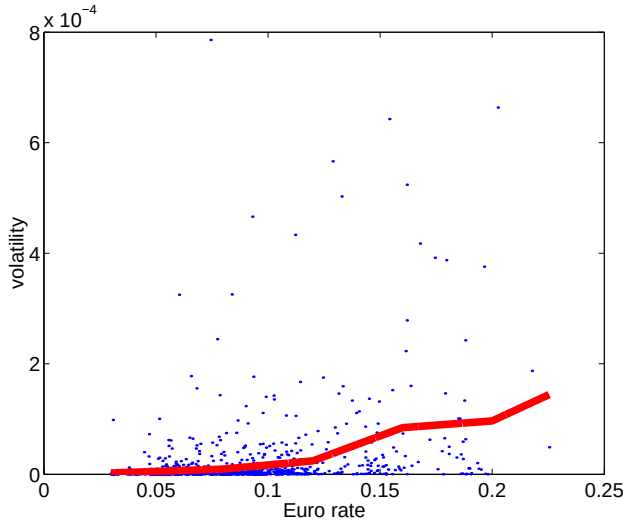

Figure 12.11: *Euro interest rate example. Linear spline estimate of the squared volatility function.*

12.6 Selecting the spline parameters

When using a spline model there are several choices to be made:

- what degree?
- how many knots?
- where to put knots?

My recommendation for the first choice is to use quadratic splines. For the third question, I recommend putting approximately an equal number

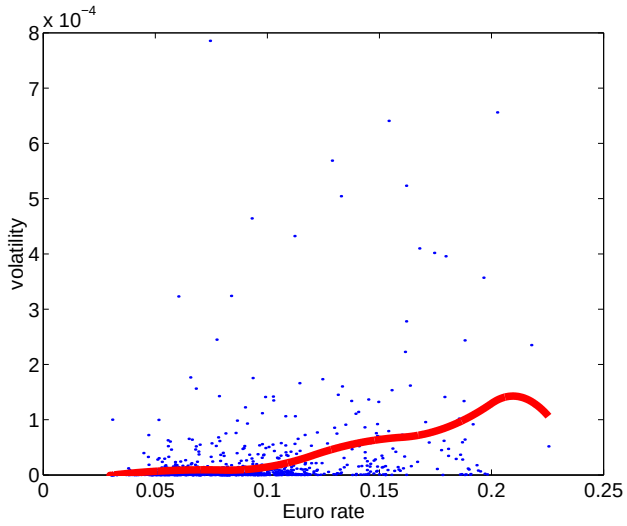

Figure 12.12: *Euro interest rate example. Quadratic spline estimate of the squared volatility function.*

of data points between the knots. For example, one could make every tenth x value a knot.

The hard question, the second one, is how many knots to use. If we fit by least-squares estimation, then the choice of how many knots is crucial. This fact can be appreciated by looking at Figure 12.13 where quadratic spline estimates of the drift function are shown for the Euro bond interest rates. Estimates with 2, 10, and 20 knots are plotted. The three splines are rather different. The spline with 20 knots seems rather wiggly and is undoubtedly overfitting the data. The 10-knot spline is also wiggly and an overfit, though not quite as much as the 20-knot spline. The spline with 2 knots seems like a reasonable estimate, though it is difficult to know for sure. The 2-knot spline doesn't bend as much on the right side as the 10 and 20-knot estimates, and that might be a problem.

One can select the number of knots by using some model selection criteria such as C_p, AIC, or SBC. To do this can fit models with 1, 2, 3, etc. knots and choose the one that minimizes C_p, say.

Another possibility is to define a number of *potential* knots and to select from among these by model selection software such as "Best subsets" in MINITAB. An implementation of this strategy is illustrated in the next

section.

So far, we have only considered estimation by least-squares, which is also called ordinary least-squares or OLS. An alternative to using model selection to choose the number of knots is to use a large, though somewhat arbitrary, number of knots and to replace least-squares estimation with an estimation method that prevents overfitting. This is done in the next section.

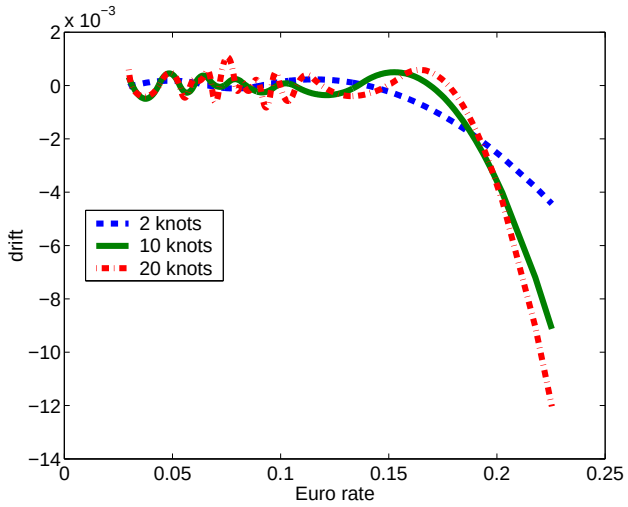

Figure 12.13: *Euro bond interest rate example. Quadratic splines estimates of the drift with 2, 10, and 20 knots fit by ordinary least-squares (OLS).*

12.6.1 Least-squares estimation in SAS

Least-squares estimation can be done easily in SAS or MINITAB. Computation of the plus functions seems easiest in SAS where one can program the computations into the "data step." In MINITAB one needs to go through a lot of menus to compute the plus functions, though MINITAB has a "macros" feature that could be used instead.

Here are some SAS programs to compute a quadratic spline estimate with *potential* knots at .08, .12, .16, and .2. The first program uses model selection by C_p to select the actual knots from the potential knots. PROC

REG is SAS's linear regression program. There are several "methods" available for the regression in PROC REG. The default method is similar to MINITAB's "regression" option.[2] In the program below, "method" is "cp." This method is very much like MINITAB's "Best subsets."[3]

```
options linesize = 64 ;
data EuroRate ;
infile 'c:\courses\or473\data\euro_rates.dat' ;
input month day year eu01 eu03 eu06;
eu01 = eu01/100 ;
diff=dif(eu01) ;
rate = lag(eu01) ;
rate2 = rate**2 ;
plus1 = ((rate - .08)**2) * (rate > .08) ;
plus2 = ((rate - .12)**2) * (rate > .12) ;
plus3 = ((rate - .16)**2) * (rate > .16) ;
plus3 = ((rate - .2)**2) * (rate > .2) ;
run ;
title 'One Month Euro dollar deposit rates' ;
proc reg ;
model diff = rate rate2 plus1 plus2 plus3 plus4 / method=cp;
run ;
```

Here is the output.

```
            One Month Euro dollar deposit rates              1
                                09:05 Thursday, April 25, 2002

                    The REG Procedure
                      Model: MODEL1
                  Dependent Variable: diff

                    C(p) Selection Method

Number in
  Model        C(p)  R-Square  Variables in Model

      2    -0.5962    0.0172  plus3 plus4
      2    -0.1048    0.0168  plus2 plus4
      1     0.0526    0.0150  plus3
      2     0.3649    0.0164  plus1 plus4
      1     0.6221    0.0145  plus4
      2     0.7035    0.0161  rate2 plus4
      2     1.0446    0.0158  plus2 plus3
```

[2]This is the subcommand called "regression" among the "regression" options of the "Stat" menu in MINITAB.

[3]This is the subcommand called "Best subsets" among the "regression" options of the "Stat" menu in MINITAB.

```
       2     1.1528    0.0157  rate plus4
       3     1.2585    0.0173  rate plus3 plus4
       3     1.2855    0.0173  rate2 plus3 plus4
       3     1.3833    0.0172  plus1 plus3 plus4
       3     1.4036    0.0172  plus2 plus3 plus4
       2     1.6279    0.0153  plus1 plus3
       3     1.8535    0.0168  plus1 plus2 plus4
       3     1.8671    0.0168  rate plus2 plus4
       3     1.8827    0.0168  rate2 plus2 plus4
(output deleted)
```

Here is a SAS program to rerun regression with the model suggested by C_P that has only a linear monomial in "rate" and only the quadratic plus functions with knots at .16 and .2.

```
(data step omitted but same as before)

proc reg ;
model diff = plus3 plus4;
output out=EuroOut p = yhat ;
run ;
proc gplot ;
plot yhat*rate ;
run ;
```

PROC REG was requested to produce an output data set containing the predicted variable as well as all the original variables, which are included by default. PROC GPLOT plots the predicted to show us the spline. The plot is Figure 12.14. Here is the SAS listing of the regression output.

```
             One Month Euro dollar deposit rates              1
                          09:05 Thursday, April 25, 2002

                      The REG Procedure
                        Model: MODEL1
                    Dependent Variable: diff

                     Analysis of Variance

                                Sum of          Mean
Source                   DF       Squares        Square   F Value

Model                     2    0.00035658    0.00017829     10.44
Error                  1196       0.02042    0.00001707
Corrected Total        1198       0.02078

                     Analysis of Variance
```

```
                  Source              Pr > F

                  Model               <.0001
                  Error
                  Corrected Total


      Root MSE              0.00413    R-Square     0.0172
      Dependent Mean    -0.00002611    Adj R-Sq     0.0155
      Coeff Var              -15829


                     Parameter Estimates

                  Parameter      Standard
Variable    DF     Estimate         Error   t Value   Pr > |t|

Intercept    1   0.00002733   0.00012091      0.23     0.8212
plus3        1     -1.45311      0.80879     -1.80     0.0726
plus4        1    -13.51699      8.29287     -1.63     0.1034
```

If one wanted to be finicky about knot placement, one could have started with more potential knots, say at .06, .07, .08, . . ., .19, .20, and .21. Then a few of these knot could be selected in the same way as we selected .16 and .2 from among .08, .12, .16, and .2.

Estimating the volatility function using SAS

The next program estimates the volatility function. First the drift function is estimated and an output data set is created that contains the residuals, which are named "resid." In a second data step the residuals are squared and the squared residuals are named "resid2." The statement "data EuroRate" starts the second data step and the statement "set EuroRate" tells SAS to use the data set "EuroRate" as input rather than reading in data from a file. Thus, "EuroRate" is used as input, updated with a new variable, and then overwritten. The net effect of both statements is to add the variable "resid2" to the data set "EuroRate."[4] Next, the squared residuals are used to estimate the volatility function. PROC REG is called again, this

[4]In some advanced SAS programming, there is a need for having several data sets in memory. If we had used the statement "data EuroRate2" in place of "data EuroRate," then the data set "EuroRate" would have been unchanged and a new data set called "EuroRate2" would have been created containing all of the variables in "EuroRate" plus "resid2."

time with "resid2" as the response and with "method=cp" to find a good model for the volatility function.

```
(Omitted statements where the
data are read in and rate2 and plus functions
created as in earlier programs.  The SAS data
set that is created is called ''EuroRate'')
title 'One Month Euro dollar deposit rates' ;
title2 'Preliminary fit to estimation drift function' ;
proc reg ;
model diff = plus3 plus4 ;
output out=EuroRate p=yhat r=resid ;
run ;
comment The next step creates the squared residuals ;
data EuroRate ;     comment Second data step ;
set EuroRate ;
resid2 = resid*resid ;
title2 'Estimation of volatility using squared residuals' ;
proc reg  ;
model resid2 = rate rate2 plus1 plus2 plus3 plus4 / method=cp ;
run ;
```

Some of the output is shown below. The model with the smallest value of C_p uses just the first two plus functions. This model implies that volatility is zero to the left of the first knot, which does not seem realistic. Therefore, I selected the model with the second smallest value of C_p which contains the linear monomial "rate" as well as the first two plus functions.

```
                    The REG Procedure
                      Model: MODEL1
                Dependent Variable: resid2

                   C(p) Selection Method

Number in
  Model        C(p)  R-Square  Variables in Model

       2     2.4838    0.1220  plus1 plus2
       3     3.1000    0.1230  rate plus1 plus2
       3     3.3554    0.1228  rate2 plus1 plus2
       3     4.1612    0.1222  plus1 plus2 plus3
       3     4.4328    0.1220  plus1 plus2 plus4
       4     4.4790    0.1234  rate rate2 plus1 plus2
       3     4.8465    0.1217  rate plus1 plus3
       2     4.8578    0.1202  rate2 plus1
       4     4.8907    0.1231  plus1 plus2 plus3 plus4
       3     4.9479    0.1216  rate2 plus1 plus3
```

```
4     4.9629    0.1231  rate plus1 plus2 plus4
4     5.0109    0.1231  rate plus1 plus2 plus3
2     5.0932    0.1201  rate plus1
2     5.1177    0.1200  rate rate2
```

The next program fits the model just selected that models the volatility function with "rate" and the first two plus functions. The fitted values are output and called "yhat_vol." Then the fitted values are plotted against "rate" to graph the estimated volatility function. The plot is Figure 12.15.

```
proc reg ;
model resid2 = rate plus1 plus2 ;
output out=EuroRate p=yhat_vol ;
run ;
proc gplot ;
plot yhat_vol*rate ;
run ;
```

The output from this program includes an analysis of variance table and estimated parameters:

```
                    The REG Procedure
                      Model: MODEL1
                Dependent Variable: resid2

                    Analysis of Variance

                               Sum of          Mean
Source                  DF       Squares        Square   F Value

Model                    3    5.37898E-7   1.792993E-7     55.86
Error                 1195    0.00000384   3.209729E-9
Corrected Total       1198    0.00000437

                    Analysis of Variance

                  Source              Pr > F

                  Model               <.0001
                  Error
                  Corrected Total


      Root MSE           0.00005665    R-Square     0.1230
      Dependent Mean     0.00001703    Adj R-Sq     0.1208
      Coeff Var           332.64060
```

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	2.659709E-8	0.00000682	0.00	0.9969
rate	1	0.00011230	0.00009543	1.18	0.2395
plus1	1	0.01213	0.00330	3.67	0.0003
plus2	1	-0.01360	0.00680	-2.00	0.0458

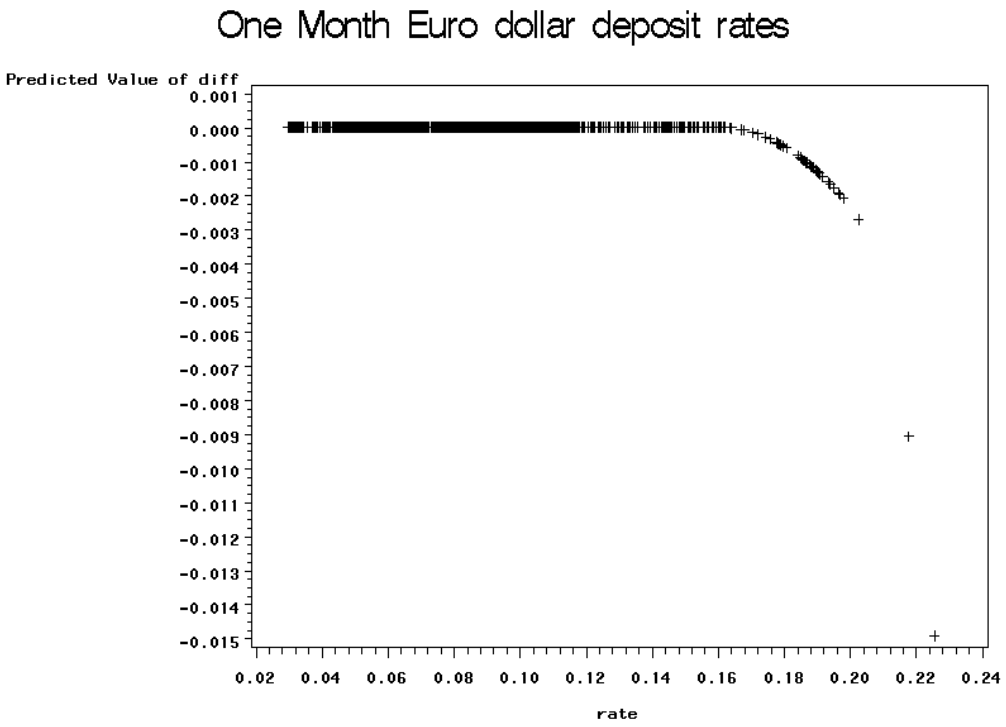

Figure 12.14: *Plot of estimated drift function from SAS.*

12.7 P-splines

Figure 12.16 shows 2, 10, and 20 knot splines estimates of the drift function by penalized least-squares, the technique that we are about to study. The

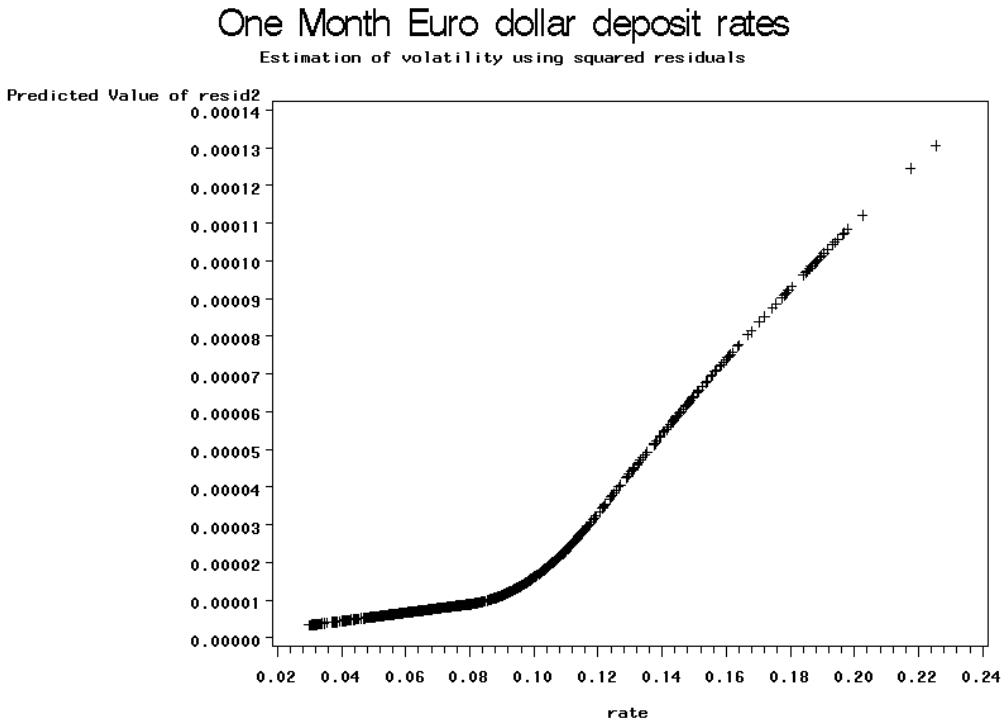

Figure 12.15: *Plot of estimated volatility function from SAS.*

estimates from penalized least-squares will be called penalized splines, or simply P-splines. Notice that the 10 and 20-knot estimates are similar, but the 2-knot estimate does not seem flexible enough to fit the data.

In this example, a spline with about 5 knots is flexible enough to fit the data. All that happens as we increase the number of knots beyond 5 is that we increase the *potential* for overfitting. However, the penalty that is being applied here prevents overfitting. Therefore, the estimated regression function does not depend greatly upon the number of knots as long as there are enough, which in this example is about 5 or more. This means we do not need to concern ourselves with the choice of number of knots. We can simply use a rather large number of knots, say 20, so that we have enough flexibility and let the penalty prevent overfitting.

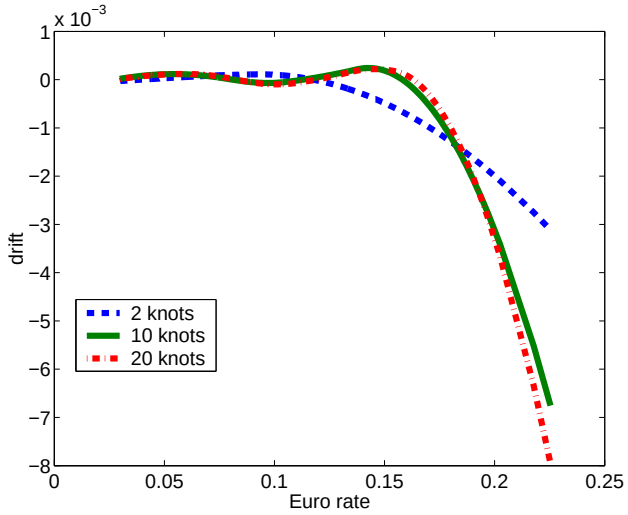

Figure 12.16: *Euro bond interest rate example. Quadratic splines estimates of the drift with 2, 10, and 20 knots by penalized least-squares (PLS). The penalty parameter was selected by GCV.*

12.7.1 Penalizing the jumps at the knots

We know that if we are using quadratic splines, then the spline's 2nd derivative jumps by an amount $2b_k$ at the kth knot. Overfitting occurs when there

are many jumps (many knots) and the jumps are unconstrained. Overfitting can be prevented by either restricting the number of jumps or the size of the jumps. Using model selection to choose the number of knots does the former. Penalization does the later.

Here's how penalization works. The sum of the squared jumps is 4 times

$$\sum_{k=1}^{K} b_k^2. \tag{12.5}$$

We wish to make small both the sum of squared errors and the sum of the squared jumps, so we add these two together with a weight λ attached to the latter. This gives us the estimation criterion

$$\sum_{i=1}^{n} \Big\{ Y_i - \Big(\beta_0 + \beta_1 x + \beta_2 x^2 + b_1 (x - t_1)_+^2 + \cdots + b_K (x - t_K)_+^2\Big)\Big\}^2 + \lambda \sum_{k=1}^{K} b_k^2. \tag{12.6}$$

The parameter λ must still be chosen. To do that, we need to understand the role that λ plays in estimation. Since λ multiplies the term (12.5) in (12.6), λ determines how much we penalize jumps in the pth derivative. If $\lambda = 0$ then there is no penalty and we are using OLS. On the other hand, as $\lambda \to \infty$, the jumps in the 2nd derivative are forced to 0. This makes the quadratic spline equal to a single quadratic polynomial. Therefore, using $\lambda = \infty$ gives the OLS fit with a quadratic *polynomial*, not a quadratic spline.

Figure 12.17 shows penalized spline fits with 25 knots and with λ equal to 0, 5, and 10^{10} (essentially ∞). We can see that the fit with $\lambda = \infty$ has little flexibility. It is a quadratic polynomial fit. The fit with $\lambda = 0$ is the OLS to a 25 knot spline already seen in Figure 12.13 to be an overfit. Using $\lambda = 5$ is a good compromise between these two extremes.

Although using 5 as the value of λ seems better than either 0 or infinity, it is natural to ask whether 5 is the best choice. In Figure 12.18 we compare the fits with λ equal to .5, 5, and 50. The three estimates of the regression function are clearly different, but it is not certain which is most appropriate. What is desirable is an automatic data-driven choice of λ. We turn to this topic next.

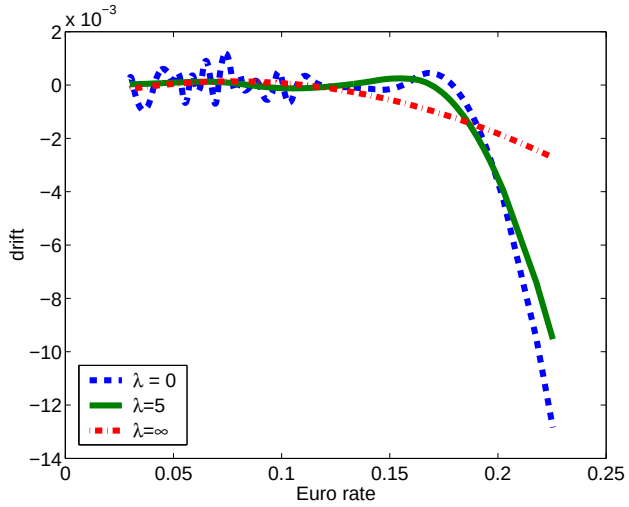

Figure 12.17: *Euro bond interest rate example. Quadratic splines estimates of the drift function, each with 25 knots, Estimation is by penalized least-squares (PLS) with* $\lambda = 0, 5,$ *and* ∞. $\lambda = 0$ *corresponds to OLS*

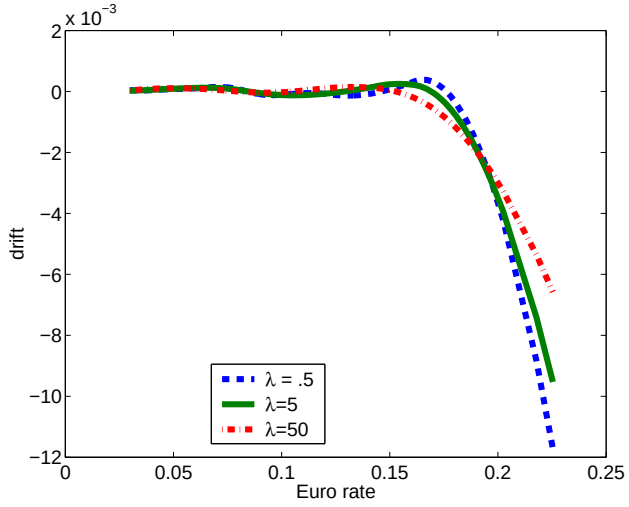

Figure 12.18: *Euro bond interest rate example. Quadratic splines estimates of the drift, each with 25 knots. Estimation is by penalized least-squares (PLS) with* $\lambda = .5, 5,$ *and 50.*

12.7.2 Cross-validation

We will use the following principle for selection of λ:

> The optimal choice of λ is the value that will give us the most accurate predictions of new data.

But can we determine how well we can predict new and as yet unobserved data? A technique called cross-validation does precisely this. Cross-validation, also called CV, is conceptually simple: to see how well using a particular trial value of λ works for prediction, we can delete one data point, estimate the regression function with the other data, and then use the estimated regression function to predict the observation that was deleted. We can repeat this procedure n times using each data point in the sample as the deleted point. This gives us n separate estimates of the expected squared prediction error using this λ.

More precisely, CV goes through the following steps for any trial value of λ:

1. For $j = 1, \ldots, n$, let $\widehat{s}(\cdot; \lambda, -j)$ be the regression function estimated using this λ and with the jth data point deleted.[5] In other words, $\widehat{s}(\cdot; \lambda, -j)$ is the spline with coefficient equal to the parameter values that minimizes

$$\sum_{i \neq j} \Big\{ Y_i - \Big(\beta_0 + \beta_1 x + \beta_2 x^2 + b_1 (x - t_1)_+^2 + \cdots + b_K (x - t_K)_+^2\Big)\Big\}^2 + \lambda \sum_{k=1}^{K} b_k^2. \tag{12.7}$$

2. Define

$$\mathrm{CV}(\lambda) = n^{-1} \sum_{j=1}^{n} \{Y_j - s(X_j; \lambda, -j)\}^2 .$$

Since $\mathrm{CV}(\lambda)$ estimates the expected squared prediction error, the value of λ that minimizes $\mathrm{CV}(\lambda)$ is considered best. The final estimate of the regression function uses all of the data and the value of λ that minimizes CV.

[5] The notation "$-j$" is intended to suggest that the jth data point has been removed or "subtracted" from the data set.

12.7.3 The effective number of parameters

If we use a pth degree spline with K knots, then there are $1+p+K$ parameters, the intercept, the p coefficients of the monomials x to x^p, and the K coefficients of the plus functions. If we use $\lambda = 0$ then all of these parameters are free to vary. However, a positive value of λ constrains the size of these last K parameters. In the extreme case where $\lambda = +\infty$, the estimated coefficients of the plus function will all be constrained to equal 0, so there are only $1+p$ free parameters.

When λ is positive but not $+\infty$, then the "effective number of parameters" should be somewhere between $1+p$ and $1+p+K$. How can we measure the effective number of parameters? The theory is somewhat complex and will not be introduced here. Only the end result will be given. Let $\boldsymbol{X}$ be the matrix

$$\boldsymbol{X} = \begin{pmatrix} 1 & X_1 & \cdots & X_1^p & (X_1-t_1)^p & \cdots & (X_1-t_K)^p \\ 1 & X_2 & \cdots & X_2^p & (X_2-t_1)^p & \cdots & (X_2-t_K)^p \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_n & \cdots & X_n^p & (X_n-t_1)^p & \cdots & (X_n-t_K)^p \end{pmatrix}. \tag{12.8}$$

Then let $\boldsymbol{D}$ be a $(1+p+K)\times(1+p+K)$ square matrix with all off-diagonal elements equal to zero and with its diagonal elements equal to $1+p$ zeros followed by K ones. Then the effective number of parameters, also called the effective degrees of freedom (DF), is

$$\text{DF}(\lambda) = \text{trace}\{(\boldsymbol{X}^\top\boldsymbol{X})(\boldsymbol{X}^\top\boldsymbol{X}+\lambda\boldsymbol{D})^{-1}\}$$

Here trace$(\boldsymbol{A})$ is the trace of the matrix $\boldsymbol{A}$ and is defined to be the sum of its diagonal elements.

You should realize that DF(λ) is generally not an integer so it is possible, for example, to have 2.5 effective parameters. To understand how this can happen, consider a linear spline with one knot which, of course, has 3 parameters. If the jump in the first derivative is constrained to be 0 (penalty is infinite), then there are only 2 effective parameters. If there is no penalty at all, then there are e effective parameters. If the jump is penalized but not completely constrained, then the number of effective parameters is somewhere strictly between 2 and 3.

Figure 12.19 is a semi-log plot of DF(λ) versus λ for a 25-knot quadratic spline ($p=2$). We can see that DF(λ) decreases from $28 = 1+p+K$ to

$3 = 1 + p$ as λ increase from 0 to ∞. CV(λ) is essentially equal to 28 if $\lambda < 10^{-8}$ and CV(λ) is essentially equal to 3 if $\lambda > 10^4$.

If $\lambda = 0$, so that we are using ordinary least squares, then DF gives us precisely the number of parameters in the model. To appreciate this, note that

$$\text{DF}(0) = \text{trace}\{(\boldsymbol{X}^\mathsf{T}\boldsymbol{X})(\boldsymbol{X}^\mathsf{T}\boldsymbol{X})^{-1}\} = \text{trace}(\boldsymbol{I}_{1_p+K}) = 1 + p + K.$$

where $\boldsymbol{I}_{1+p+K}$ is the $(1 + p + K) \times (1 + p + K)$ identity matrix.

DF can be used to get a "corrected" and nearly unbiased estimate of σ^2. Let $\widehat{s}(\cdot; \lambda)$ be the estimated regression function using *all* the data. Then

$$\widehat{\sigma}^2(\lambda) = \frac{\sum \{Y_i - \widehat{s}(X_i; \lambda)\}^2}{n - \text{DF}(\lambda)}. \tag{12.9}$$

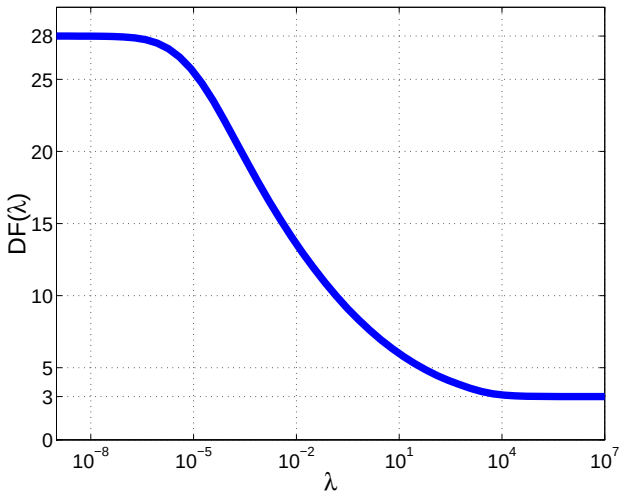

Figure 12.19: *Euro bond interest rate example. Estimation of the drift function with 25-knot quadratic splines. Plot of DF as a function of* λ.

12.7.4 Generalized cross validation

Computing CV(λ) is somewhat time consuming, though there are some tricks that are used that make the computations easier than they would

seem.[6] However, there is an approximation to CV that is very quick to compute, faster than CV. The generalized cross validation statistic (GCV) is

$$\text{GCV}(\lambda) = \frac{n^{-1}\sum\{Y_i - \widehat{s}(X_i;\lambda)\}^2}{\left(1 - \frac{\text{DF}(\lambda)}{n}\right)^2}.$$

The point is that GCV only uses the estimate $\widehat{s}$ computed from all of the data.

Figure 12.20 is a semi-log plot of GCV(λ) versus λ. We can see that GCV is minimized by λ somewhere near 10. In fact, the minimum occurs at $\lambda = 13.3$. Figure 12.21 is a plot of GCV(λ) versus DF(λ). This is essentially the same function as in curve 12.20 except that the horizontal scale is DF(λ) rather than λ. Notice that GCV is minimized by approximately 6 effective parameters. In fact, the minimum is at 5.8 DF.

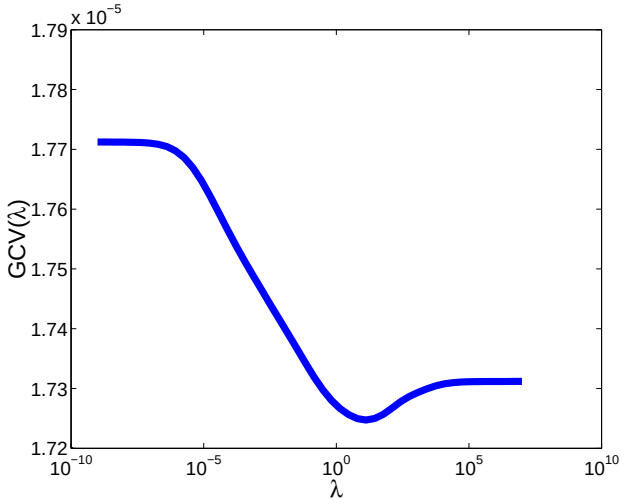

Figure 12.20: *Euro bond interest rate example. Estimation of the drift function with 25-knot quadratic splines. Semilog plot of GCV versus λ.*

[6] In particular, one does not really need to compute n separate regressions. There are formulas that show the effect of deleting an observation without actually needing to recalculate the estimate with the observation deleted.

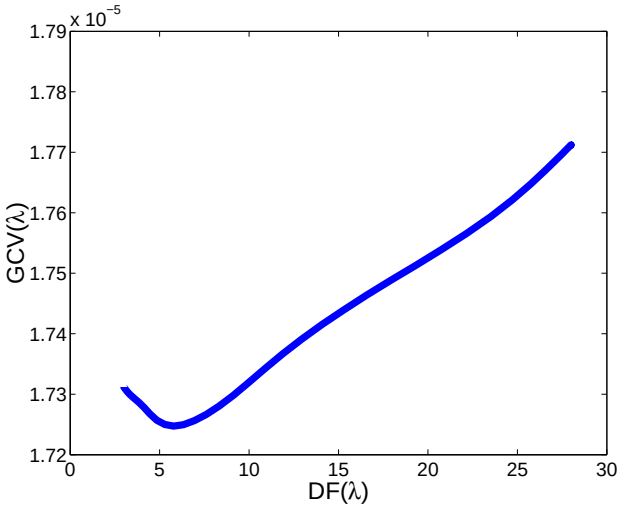

Figure 12.21: *Euro bond interest rate example. Estimation of the drift function with 25-knot quadratic splines. Plot of GCV versus DF. GCV selects approximately 6 degrees of freedom.*

12.7.5 AIC

For linear regression models, AIC is

$$\text{AIC} = n \log 9\widehat{\sigma}^2) + 2(1+p)$$

where $1+p$ is the number of parameters in a model with p predictor variables; the intercept gives us the final parameter. Since DF(λ) is the effective number of parameter of a P-spline, we can define AIC for P-splines as

$$\text{AIC}(\lambda) = n \log\{\widehat{\sigma}^2(\lambda)\} + 2\text{DF}(\lambda).$$

We can then select λ by minimizing AIC. Using AIC to select λ usually gives a similar estimate as using CV or GCV. In fact, it has been shown theoretically that all three criteria should give similar estimates. Figure 12.22 is a plot of AIC versus DF(λ). Notice that the curve has a similar shape to the one in Figure 12.21 and the minimum in both figures occurs around DF $= 6$.

Why do we have so many criteria for selecting the penalty parameter? The answer is that this is mostly a historical accident. AIC was developed

by time series analysts, CV arose in parametric statistics, and GCV was introduced in nonparametric regression, specifically in spline estimation. Three different groups of researchers were attacked the same general problem and produced three similar, though not identical, answers. In fact, there were four similar answers, since C_p is similar to CV, GCV, and AIC.

AIC and GCV can both be computed very quickly and usually give essentially the same answers, and then it really does not matter which is used. CV gives similar answers to AIC and GCV but is a bit more work to compute, so I do not recommend CV. I use GCV because I have a lot of experience using GCV and have found it to be reliable.

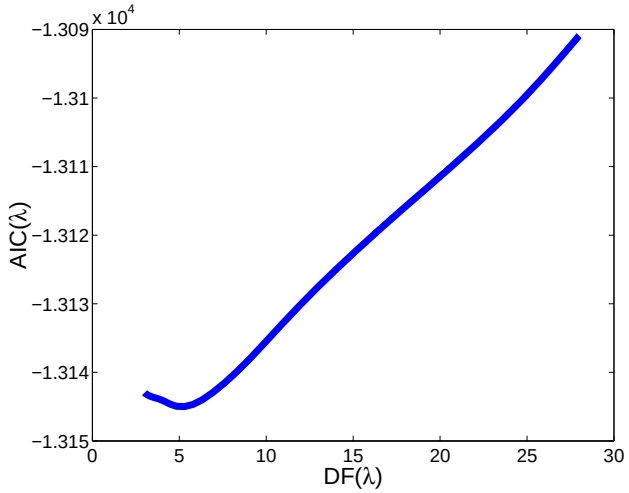

Figure 12.22: *Euro bond interest rate example. Estimation of the drift function with 25-knot quadratic splines. Plot of AIC versus DF(λ). AIC selects approximately 6 degrees of freedom, the same as GCV.*

12.8 MATLAB program "PsplineDR03.m"

The course web site has three MATLAB programs for computing P-spline estimates:

PsplineDR03.m The main program. Computes P-spline estimates with the penalty parameter chosen by GCV.

powerbasis01.m Called by "PsplineDR03.m." Given the knots and the degree of the spline, computes the monomials and plus functions.

quantileknots.m Called by "powerbasis01.m." Finds knot locations that put, as nearly as possible, equal numbers of x values between knots.

"PsplineDR03.m" calls "powerbasis01.m" which calls "quantileknots.m." For applications, you only need to understanding "PsplineDR03.m." The use of the other two programs goes on in the background. The call to "PsplineDR03.m" is simply

```
outputname = PsplineDR03(x,y,param)
```

"param" can be replaced by any name you like and names a structure containing *optional* input such as the degree of the spline, the number of knots, and "penwt" which is a grid of λ values on which GCV is computed. There are several examples of "param" given below. For most applications, you can use the default values in which case "param" can be omitted so that the call is

```
outputname = PsplineDR03(x,y)
```

A list of all optional input that can be specified by "param" is given in the listing of the program.

"outputname" can be any name you wish and names a structure containing all of the output including the following:

- degree = degree of the spline
- knots = knots of the spline
- penwt = values of the penalty parameter used to search for the minimum of GCV
- imin = index of value of penwt that minimizes GCV, so, for example, if imin is 14 then the 14th value of penwt is where the minimum occurs
- yhat = fitted values = regression function estimate at each x value

- beta = regression coefficients in order: intercept, monomial coefficients, plus function coefficients
- ulimit = upper 95% confidence limits of regression function at each x value
- llimit = lower 95% confidence limits of regression function at each x value
- yhatder = estimate of the derivative of the regression function at each x value
- ulimitder = upper 95% confidence limits of the derivative of the regression function at each x value
- llimitder = lower 95% confidence limits of the derivative of the regression function at each x value
- xgrid = fine grid of x values (not the values in the data)
- mhat = estimate of regression function evaluated on xgrid

The "DR" in "PsplineDR03.m" stands for Demmler-Reinsch, the inventors of an algorithm that makes computation of the Pspline at all values of "penwt" very fast.

The easiest way to use "PsplineDR03.m" put "PsplineDR03.m," "powerbasis01.m," and "quantileknots.m." in a directory along with the data and your program that load the data and calls "PsplineDR03.m."

The following MATLAB code that in part of a program that estimates the drift and volatility functions and their first derivative for the Euro deposit rates. The complete program is called "euro_interest.m" and is on the course web site. The data are read by the first line of code. After some data manipulation which is omitted here, "rate" is the interest rate while "diff" is the first differences of "rate."

```
load Euro_hw.dat ;

(code to manipulate data omitted)

fit = PsplineDR03(rate,diff,struct('nknots',25) ) ;

subplot(2,1,1) ;
```

```
p = plot(rate,fit.yhat, ...
rate,fit.ulimit,'--', ...
rate,fit.llimit,'--', ...
[0 .25], [0 0]) ;

(code omitted that enhances the plot)

subplot(2,1,2) ;

p=plot(rate,fit.yhatder, ...
rate,fit.ulimitder,'--', ...
rate,fit.llimitder,'--', ...
[0 .25], [0 0]) ;

(code omitted that enhances the plot)
```

Here "outputname" is "fit."

The number of knots is specified as 25 rather than the default value. The degree of the spline will be 2, the default value.

In the first subplot, the code

```
rate,fit.yhat
```

in the "plot" command plots the regression function and

```
rate,fit.ulimit,'--'
rate,fit.llimit,'--', ...
```

plots the upper and lower confidence limits.

The cod "[0 .25],[0 0]" puts a horizontal line through 0 by plotting the points (0,0) and (.25,0) and connecting them by a straight line.

In the second subplot, the estimated derivative and its confidence limits are plotted. The plots of the drift function and its derivative are in Figure 12.23.

Further in the program, the code

```
fit_vol = PsplineDR03(rate,(diff-fit.yhat).^2, ...
struct('nknots',25,'degree',1) ) ;
```

estimates the volatility function by regressing the squared residuals

```
(diff-fit.yhat).^2
```

on "rate." Notice the use of the name "fit_vol" to distinguish the output of this regression from the output of the earlier regression. Also, a linear

rather than a quadratic spline is being used here. Linear and quadratic splines give similar, though not identical estimates. The main purpose behind using the linear spline is to show what a linear P-spline looks like. The plots of the volatility function and its derivative are in Figure 12.24. No kinks are visible in the estimate. The kinks are there since the estimate is a linear spline, but the penalty keeps them so small that they are not noticeable. The derivative of the estimate is a 0 degree spline. The "steps" in this function are small but visible. The knot locations are clearly visible. Notice that the knots become less dense as "rate" increases because higher rates are rarer.

The drift function is very close to 0 for "rate" less than .16. For larger values of "rate" the estimated drift function is negative. However, there are very little data with "rate" greater than .16, so a natural question is whether the negative estimate of the drift function for "rate" greater than .16 is something real or instead due to estimation error. This question can be answered by the confidence intervals. Notice that on the far right the confidence intervals lie entirely below zero, showing that with 95% confidence the drift function is negative.

Notice that the confidence bands are wider near the boundaries of the data and the confidence bands around the derivative of the estiamte are wider than around the estimate itself.

12.9 Additive Models

So far we have only considered spline modeling with a single predictor variable. Now let us assume that there are two predictors, x_1 and x_2. The most general nonparametric model is

$$Y_i = \mu(X_{1,i}, X_{2,i}) + \epsilon_i,$$

where $\mu(x_1, x_2)$ is a completely arbitrary function of x_1 and x_2. However, developing splines models with this level of generality is beyond the scope

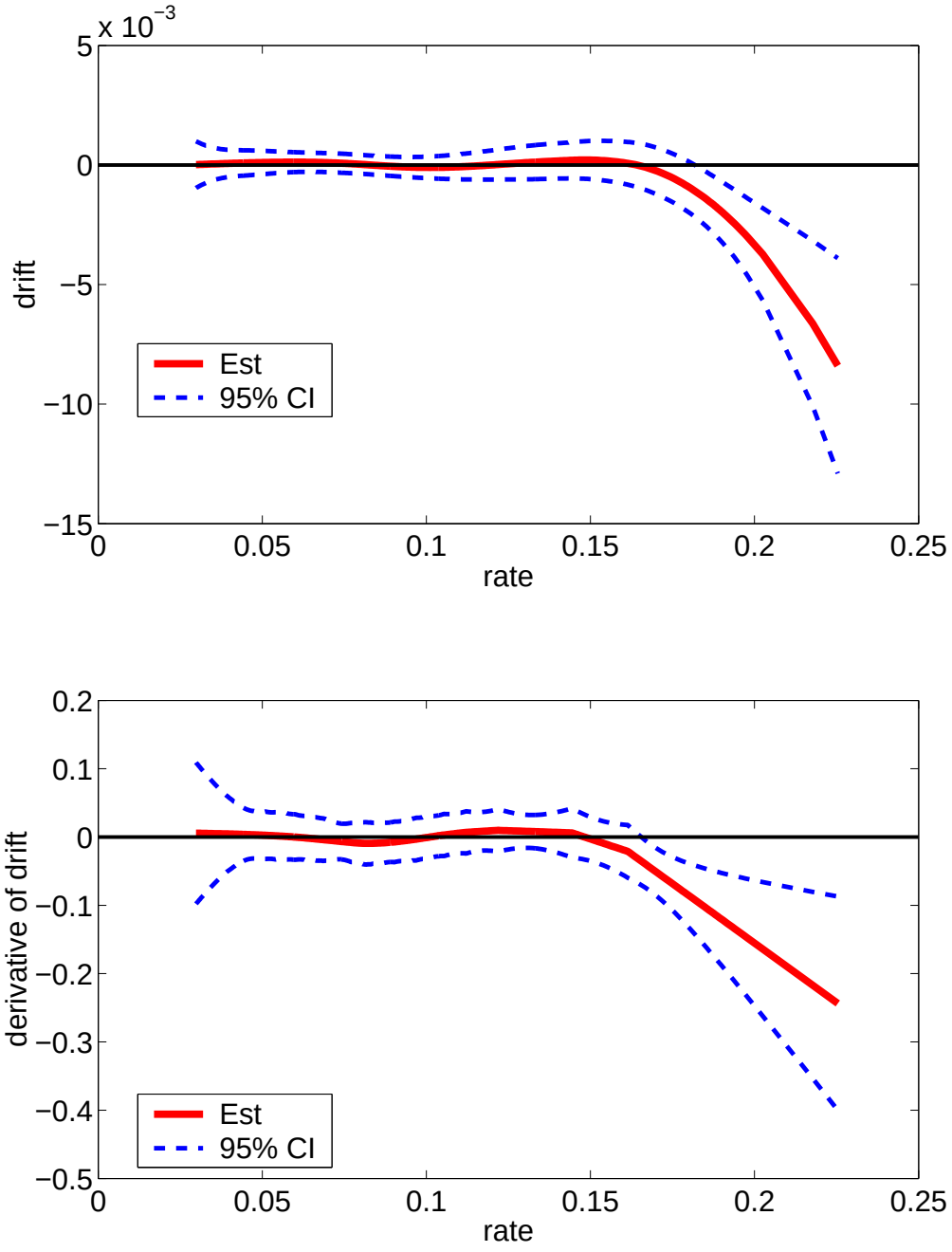

Figure 12.23: Top: *Estimate of the drift function with 95% confidence intervals. The estimate is a 25-knot quadratic P-spline with the penalty parameter estimated by GCV.* Bottom: *First derivative of the estimated drift function with 95% confidence intervals.*

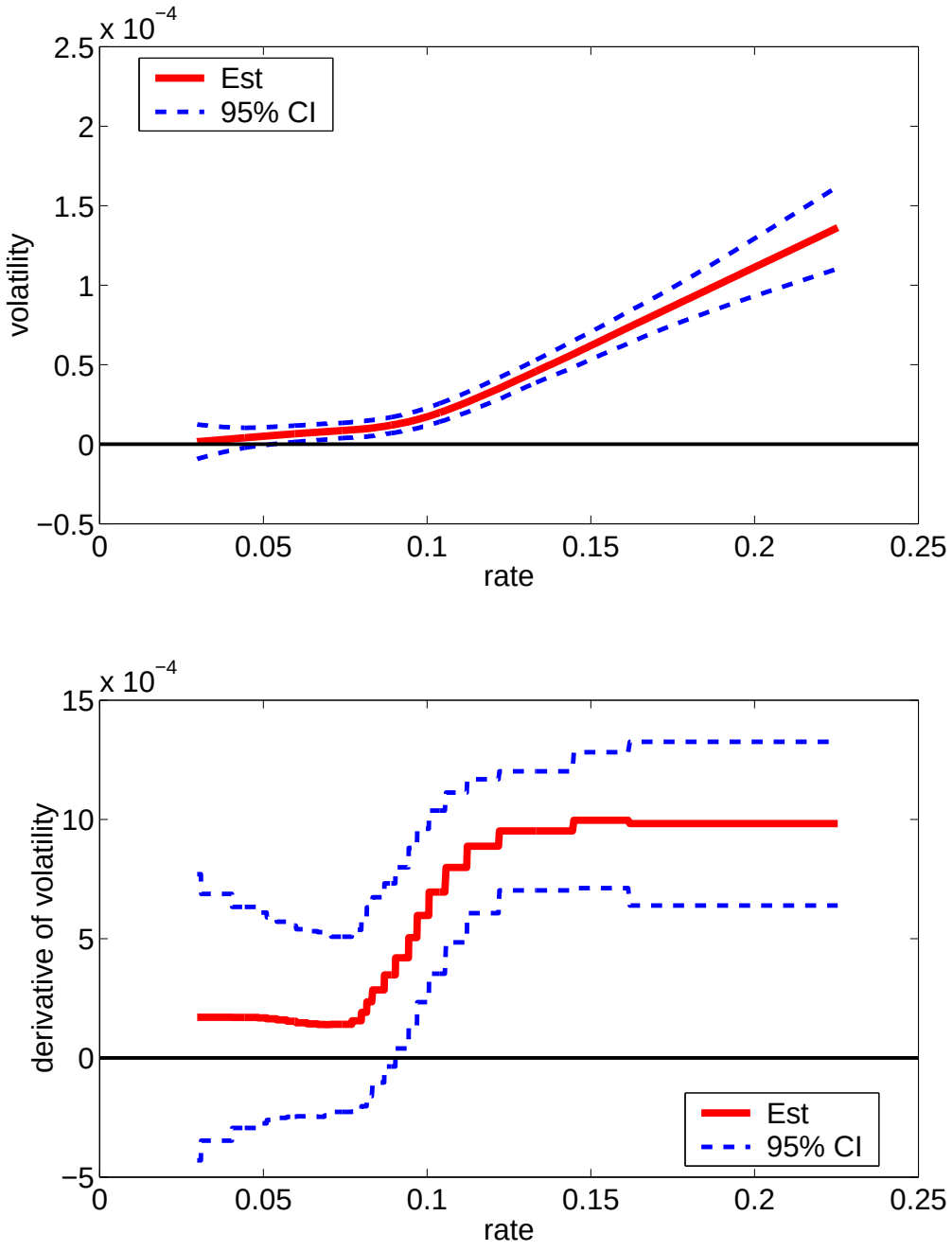

Figure 12.24: Top: *Estimate of the volatility function with 95% confidence intervals. The estimate is a 25-knot linear P-spline with the penalty parameter estimated by GCV.* Bottom: *First derivative of the estimated volatility function with 95% confidence intervals.*

of this course. We will instead use a simpler type of model called an additive model:

$$Y_i = \mu_1(X_{1,i}) + \mu_2(X_{2,i}) + \epsilon_i.$$

Here $\mu_1(x_1)$ is a function of x_1 only and similarly for $\mu_2(x_2)$. An additive spline model assumes that μ_1 and μ_2 are splines. Additive spline models are easy to fit in, for example, SAS. One just constructs monomials and plus functions in x_1 and monomials and plus functions in x_2.

The next programs fits an additive model to the Euro dollar one-month rates. The predictors are "rate" and "lagdiff" which is the lagged value of the response "diff." The first program searches for a good model using C_p.

```
options linesize = 64 ;
data EuroRate ;
infile 'c:\courses\or473\data\euro_rates.dat' ;
input month day year eu01 eu03 eu06;
eu01 = eu01/100 ;
diff=dif(eu01) ;
rate = lag(eu01) ;
lagdiff = lag(diff) ;
rate2 = rate**2 ;
lagdiff2 = lagdiff**2 ;
plus1 = ((rate - .08)**2) * (rate > .08) ;
plus2 = ((rate - .12)**2) * (rate > .12) ;
plus3 = ((rate - .16)**2) * (rate > .16) ;
plus4 = ((rate - .2)**2) * (rate > .2) ;
plusLD1 = ((lagdiff - 0)**2) * (lagdiff > 0) ;
run ;
title 'One Month Euro dollar deposit rates' ;
title2 'Additive Model' ;
proc reg  ;
model diff = rate rate2 plus1 plus2 plus3 plus4
lagdiff lagdiff2 plusLD1 / method=cp ;
run ;
```

Here is the listing.

```
One Month Euro dollar deposit rates              1
           Additive Model
                    18:01 Thursday, April 25, 2002

          The REG Procedure
            Model: MODEL1
      Dependent Variable: diff

        C(p) Selection Method
```

```
Number in
  Model        C(p)  R-Square  Variables in Model

       4     0.8096    0.0623  plus2 plus3 lagdiff plusLD1
       3     1.1730    0.0604  plus4 lagdiff plusLD1
       4     1.1833    0.0620  plus3 plus4 lagdiff plusLD1
       3     1.3698    0.0603  plus3 lagdiff plusLD1
       4     1.4818    0.0617  plus1 plus3 lagdiff plusLD1
       4     2.2736    0.0611  plus2 plus4 lagdiff plusLD1
       5     2.4572    0.0626  plus2 plus3 plus4 lagdiff plusLD1
       4     2.5082    0.0609  rate2 plus3 lagdiff plusLD1
       5     2.6407    0.0624  rate2 plus2 plus3 lagdiff plusLD1
       5     2.6446    0.0624  plus1 plus3 plus4 lagdiff plusLD1
       5     2.6669    0.0624  rate plus2 plus3 lagdiff plusLD1
       5     2.6790    0.0624  plus2 plus3 lagdiff lagdiff2
                               plusLD1
(output omitted)
```

As an illustration, the following program fits the model with the three variables "plus4," "lagdiff," and "plusLD1" that has the second lowest C_p value.[7]

```
(data read in and manipulated as in previous program)

proc reg ;
model diff = plus4 lagdiff plusLD1;
output out=EuroOut p =yhat ;
run ;
```

```
                      The REG Procedure
                        Model: MODEL1
                  Dependent Variable: diff

                     Analysis of Variance

                                  Sum of          Mean
Source                 DF       Squares        Square   F Value

Model                   3       0.00125    0.00041818     25.59
Error                1194       0.01951    0.00001634
Corrected Total      1197       0.02077

                     Analysis of Variance

                  Source              Pr > F

                  Model               <.0001
```

[7] One might, of course, prefer the model with the lowest C_p values, which is a four variable model.

```
                 Error
                 Corrected Total


      Root MSE              0.00404    R-Square     0.0604
      Dependent Mean    -0.00002346    Adj R-Sq     0.0580
      Coeff Var              -17235


                     Parameter Estimates

                 Parameter      Standard
Variable    DF      Estimate         Error   t Value   Pr > |t|

Intercept    1    0.00013987    0.00012053      1.16     0.2461
plus4        1     -19.96184       5.96808     -3.34     0.0008
lagdiff      1       0.27061       0.03544      7.64     <.0001
plusLD1      1     -16.79625       3.63164     -4.62     <.0001
```

The following SAS program uses the parameter estimates from this listing to compute the estimates of μ_1 and μ_2 and plot the estimate of μ_1.

```
options linesize = 64 ;
data EuroRate ;
infile 'c:\courses\or473\data\euro_rates.dat' ;
input month day year eu01 eu03 eu06;
eu01 = eu01/100 ;
diff=dif(eu01) ;
rate = lag(eu01) ;
lagdiff = lag(diff) ;
plus4 = ((rate - .2)**2) * (rate > .2) ;
plusLD1 = ((lagdiff - 0)**2) * (lagdiff > 0) ;
mu1= -19.96184 * plus4 ;
mu2 = .27061*lagdiff - 16.79625 * plusLD1 ;
run ;
title 'One Month Euro dollar deposit rates' ;
proc gplot ;
plot mu1*rate ;
run ;
```

To plot μ_2 change the last three lines of this program to:

```
proc gplot ;
plot mu2*lagdiff ;
run ;
```

The SAS plots are Figures 12.25 and 12.26. A MATLAB plot of μ_2 is shown in Figure 12.27.

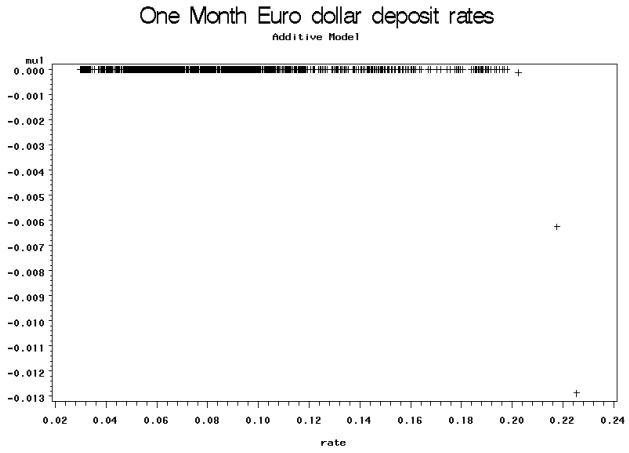

Figure 12.25: *Estimate of μ_1. SAS plot.*

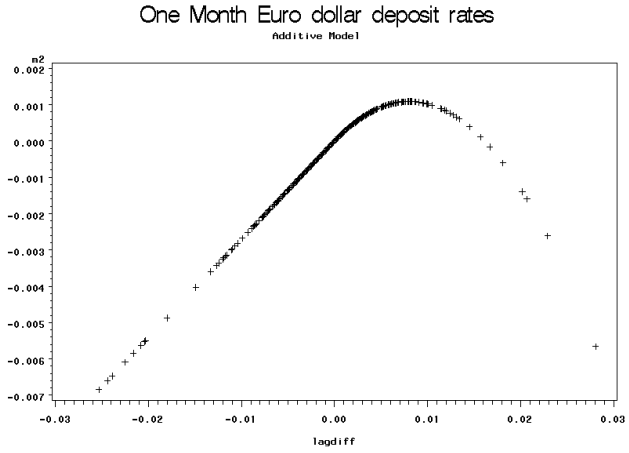

Figure 12.26: *Estimate of μ_2. SAS plot.*

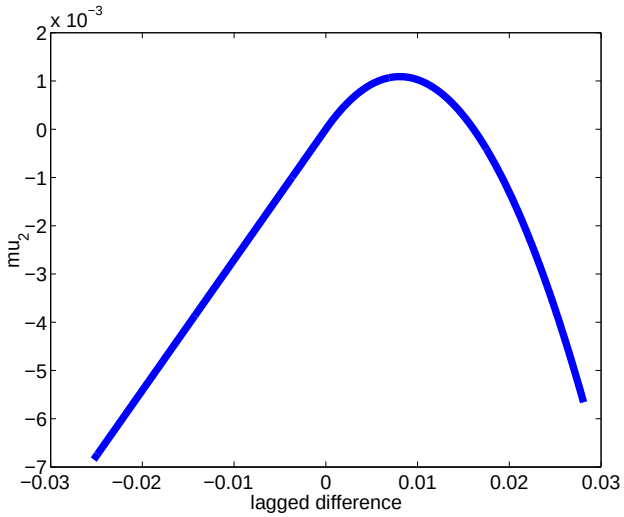

Figure 12.27: *Estimate of μ_2. MATLAB plot.*

12.10 References

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Chapter 13

Behavioral finance: 5/1/02

13.1 Introduction

- behavioral finance is an alternative to the EMH
- this material taken mostly from the 2000 book by Andrei Shleifer of Harvard:
  - **Inefficient Markets: An Introduction To Behavioral Finance**
- EMH has been the central tenet of finance for almost 30 years
- power of the EMH assumption is remarkable
- EMH started in the 1960's
  - immediate success in theory and empirically
  - early empircal work gave overwhelming support to EMH
  - EMH invented at Chicago and Chicago became a world center of research in finance
  - Jensen (1978) "no other proposition in economics . . . has more solid empirical support"
- verdict is changing
  - efficiency of arbitrage is much weaker than expected
  - true arbitrage possibilities are rare

  - near arbitrage is riskier than expected
  - "Markets can remain irrational longer than you can remain solvent" — John Maynard Keyes
    * quoted by Roger Lowenstein in **When Genius Failed: The Rise and Fall of Long-Term Capital Management**

13.2 Defense of EMH

- three lines of defense of the EMH:
  - investors are rational
  - trading of irrational investors is random and their trades cancel each other
  - even if a "herd" of irrational investors trade in similar ways, rational arbitrageurs will eliminate their influence on market price
- each of these defenses is weaker that had been thought
- rational investing = "value a security by its fundamental value"
  - "fundamental value" = net present worth of all future cash flows
- rational investing $\Rightarrow$ prices are (geometric) random walks
- but prices being random walks (or nearly so) does **not** imply rational investing
- there is good evidence that irrational trading is correlated
  - look at the internet stock bubble
- initial tests of the semi-strong form of efficiency supported that theory
  - event studies showed that the market did react immediately to news and then stopping reactin
    * so reaction to news, as EMH predictos
    * also no reaction to stale news, again as EMH predicts
  - Scholes (1972) found little reaction to "non news"
    * block sales had little effect on prices

13.3 Challenges to the EMH

- it is difficult to maintain that all investors are rational.
  - many investors react to irrelevant information
  - Black calls them noise traders
- investors act irrationally when they
  - fail to diversify
  - purchase actively and expensively managed mutual funds
  - churn their portfolios
- investors do not look at final levels of wealth when assessing risky situations ("prospect theory")
- there is a serious "loss aversion"
- people do not follow Bayes rule for evaluating new information
  - too much attention is paid to recent history
- overreaction is commonplace
- these deviations from fully rational behavior are **not random**
- moreover, noise traders will follow each others mistakes
- thus, noise trading will be correlated across investors
- managers of funds are themselves human and will make these errors too
- managers also have their own types of errors
  - buying portfolios excessively close to a benchmark
  - buying the same stocks as other fund managers (so as not to look bad)
  - window dressing — adding stocks to the portfolio that have been performing well recently

  - on average, pension and mutual fund managers underperform passive investment strategies
    * these managers might be noise traders too

13.4 Can arbitrageurs save the day?

- the last defense of the EMH depends on arbitrage
- even if investor sentiment is correlated and noise traders create incorrectly priced assets
  - arbitrageurs are expected to take the other side of these traders and drive prices back to fundamental values
- a fundamental assumption of behavioral finance is that real-world arbitrage is risky and limited
- arbitrage depends on the existence of "close substitutes" for assets whose prices have been driven to incorrect levels by noise traders
- many securities do not have true substitutes
- often there are no risk-less hedges for arbitrageurs
- mispricing can get even worse, as the managers of LTCM learned
  - this is called noise trader risk

13.5 What do the data say?

- Shiller (1981), "Do stock prices move too much to be justified by subsequent changes in dividends":
  - market prices are too volatile
  - more volatile than explained by a model where prices are expected net present values
  - this work has been criticized by Merton who said that Shiller did not correctly specify fundamental value

- De Bondt and Thaler (1985), "Does the stock market overreact?":
  - frequently cited and reprinted paper
  - work done at Cornell
  - compare extreme winners and losers
  - find strong evidence of overreaction
  - for every year starting at 1933 they formed portfolios of the best performing stocks over the previous three years
    - "winner portfolios"
  - they also formed portfolios of the worse performing stocks
    - "loser portfolios"
  - then examined returns on these portfolios over the next five years
    - losers consistently outperformed winners
  - difference is difficult to explain as due to differences in risk, at least according to standard models such as CAPM
  - De Bondt and Thaler claim that investors overreact
    - extreme losers are too cheap
    - so they bounce back
  - the opposite is true of extreme winners
- historically, small stocks have earned higher returns than large stocks
  - no evidence that the difference is due to higher risk
  - superior returns of small stocks have been concentrated in January
  - small firm effect and January effect seem to have disappeared over the last 15 years
- market to book value is a measure of "cheapness"
  - high market to book value firms are "growth" stock
    - they tend to underperform
    - also they tend to be riskier, especially in severe down markets

- October 19, 1987 — Dow Jones index dropped 22.6%
  - there was no apparent news that day
- Cutler et al (1991): looked at 50 largest one-day market changes
  - many came on days with no major news announcements
- Roll (1988) tried to predict the share of return variation that could be explained by
  - economic influences
  - returns on other stocks in the same industry
  - public firm-specific news
- Roll's findings:
  - $R^2 = .35$ for monthly data
  - $R^2 = .2$ for daily data
- Roll's study also shows that there are no "close substitutes" for stocks
  - this lack of close substitutes limits arbitrage
- stocks rise if the company is put on the S&P 500 index
  - this is reaction to "non news"
  - America Online rose 18% when included on the S&P
- In summary, there is now considerable evidence against the EMH
  - This evidence was not found during early testing of the EMH
  - Researchers needed to know what to look for

13.6 Market Volatility and Irrational Exuberance

- Two books by Robert J. Shiller:
  - 1989 — "Market Volatility"
  - 2000 — "Irrational Exuberance"

- What is a stock worth?
- Let V_t be intrinsic value at time t.
- By definition

$$V_t = \frac{C_{t+1}}{1+k} + \frac{C_{t+2}}{(1+k)^2} + \cdots = \sum_{i=1}^{\infty} \frac{C_{t+i}}{(1+k)^i}.$$

  - C_k is the cash flow at time k
  - k is the discount rate

- A little bit of algebra shows that

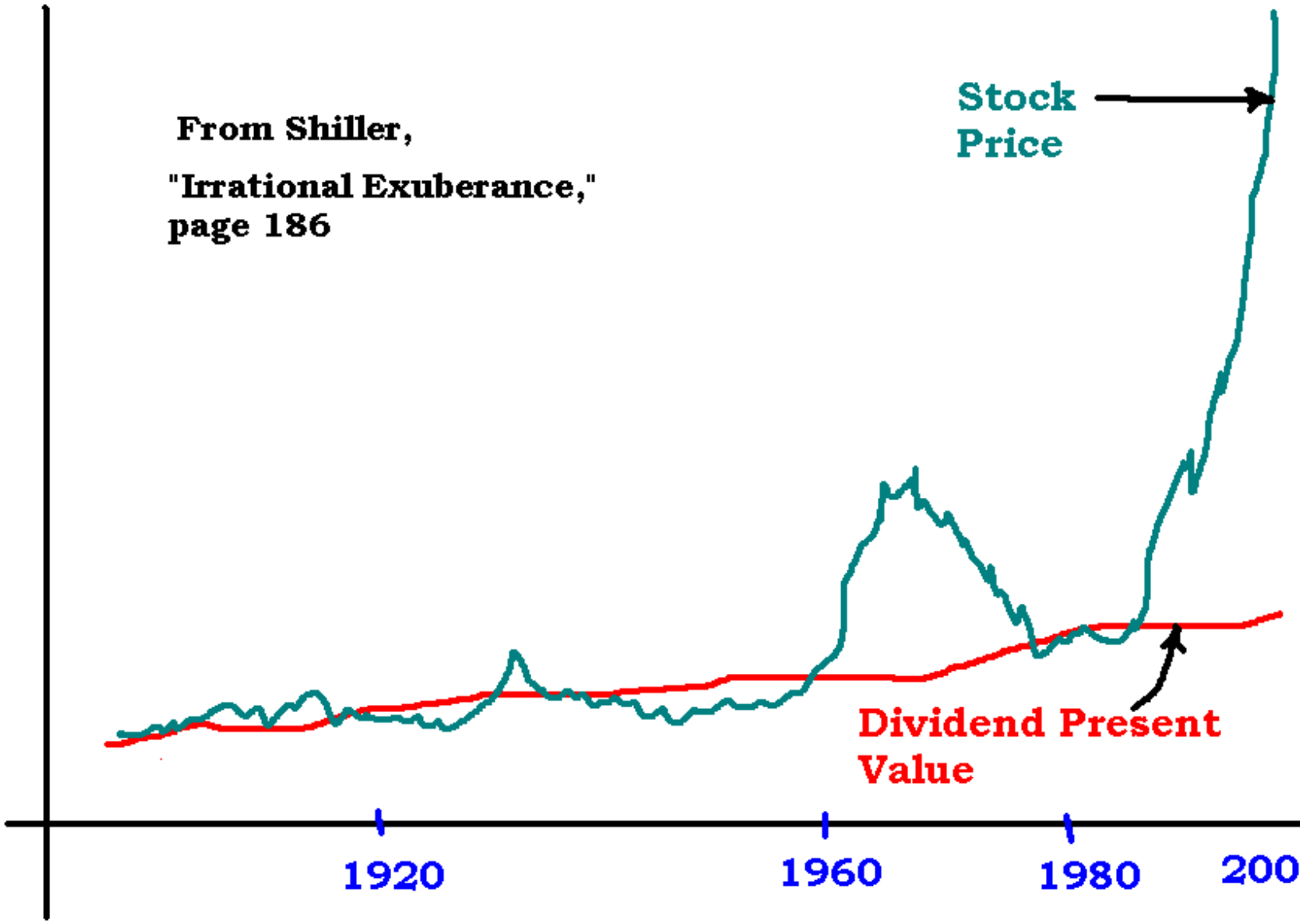
$$V_t = \sum_{i=1}^{T} \frac{C_{t+i}}{(1+k)^i} + \frac{V_T}{(1+k)^T}.$$

  (exercise: check)

- From previous page:

$$V_t = \sum_{i=1}^{T} \frac{C_{t+i}}{(1+k)^i} + \frac{V_T}{(1+k)^T}. \tag{13.1}$$

- **Shiller's idea:**
  - $T =$ now
  - $t < T$ is "the past"
  - Use past data to compute V_t
    - as an approximation use P_T in place of V_T
    - then all quantities in (13.1) are known at time T
  - compare V_t with $P_t =$ actual stock price
- **EMH says that P_t is the optimal forecast at time t of V_t.**
- How do V_t and P_t compare?



- We see that P_t is **more** volatile than V_t.
- Does this agree with the hypothesis that P_t is the optimal forecast of V_t?

  NO!

13.6.1 Best prediction

- If $\widehat{X}$ is the best predictor of X, then
  - $\widehat{X}$ and $X - \widehat{X}$ are uncorrelated
  - $\text{Var}(X) = \text{Var}(\widehat{X}) + E(X - \widehat{X})^2$.
  - $\text{Var}(\widehat{X}) \leq \text{Var}(X)$.

  (Exercise: prove these results if $\widehat{X}$ is the best linear predictor of X.)
- **An optimal predictor is less variable than the quantity being predicted.**

Example: random walk

Suppose that $W_t = \epsilon_1 + \cdots + \epsilon_t$, where $\epsilon_1, \ldots$ are IID $N(\mu, \sigma^2)$.

- at time t, $0 \leq t < T$, the best predictor of W_T is $\widehat{W}_T = W_t + (T - t)\mu$
- In other words, at time t

  $$W_T = \{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t\} + \{\epsilon_{t+1} + \cdots + \epsilon_T\}$$

  is predicted by

  $$\widehat{W}_T = \{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t\} + \{\mu + \cdots + \mu\}$$

- From last page:

  $$W_T = \{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t\} + \{\epsilon_{t+1} + \cdots + \epsilon_T\}$$

  is predicted by

  $$\widehat{W}_T = \{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t\} + \{\mu + \cdots + \mu\}$$

- $\text{Var}(W_T) = T\sigma^2$
- $\text{Var}(\widehat{W}_T) = \text{Var}(W_t) = t\sigma^2$.
- $W_T - \widehat{W}_T = (\epsilon_{t+1} - \mu) + \cdots + (\epsilon_T - \mu)$
- $\text{Var}(W_T - \widehat{W}_T) = (T - t)\sigma^2$.

Therefore, in this example,

$$\text{Var}(W_T) = \text{Var}(\widehat{W}_T) + \text{Var}(W_T - \widehat{W}_T)$$

As $t \uparrow T$ we cumulate more information about W_T and

- $\text{Var}(\widehat{W}_T) = \text{Var}(W_t) = t\sigma^2$ increases
- $\text{Var}(W_T - \widehat{W}_T) = (T-t)\sigma^2$ decreases
- $\text{Var}(W_T) = T\sigma^2$ stays the same (of course)

The main point, however, is simply that

- an optimal forecast is less variable that what is being forecasted.
- stock prices are **more** variable that the present values of future discounted dividends.

Therefore, price cannot be optimal forecasts of the present value of discounted future dividends.

At least, this is Shiller's argument.

- "Irrational Exuberance" is a very interesting discussion of market psychology, bubbles, "naturally occurring Ponzi schemes," and other possible explanations of why the market seemed overpriced in 2000.
- Shiller's analysis suggests that the market may be still overpriced in 2002, despite two years of declining prices.
- Shiller presents fascinating evidence that periods where the market is either over or under priced have occurred, often several times, in many countries.

13.7 References

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